TERMINOLOGY: The sum of the diagonal elements of a square matrix $\mathbf{A}$ is called the **trace** of $\mathbf{A}$, written $tr(\mathbf{A})$, that is, for $\mathbf{A}_{n\times n} = (a_{ij})$,

$$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

Result MAR6.1.

1. $tr(\mathbf{A} \pm \mathbf{B}) = tr(\mathbf{A}) \pm tr(\mathbf{B})$
2. $tr(c\mathbf{A}) = ctr(\mathbf{A})$
3. $tr(\mathbf{A}') = tr(\mathbf{A})$
4. $tr(\mathbf{AB}) = tr(\mathbf{BA})$
5. $tr(\mathbf{A}'\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2$.

TERMINOLOGY: The **determinant** of a square matrix $\mathbf{A}$ is a real number denoted by $|\mathbf{A}|$ or $det(\mathbf{A})$.

Result MAR6.2.

1. $|\mathbf{A}'| = |\mathbf{A}|$
2. $|\mathbf{AB}| = |\mathbf{BA}|$
3. $|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}$
4. $|\mathbf{A}| = 0$ iff $\mathbf{A}$ is singular
5. For any $n \times n$ upper (lower) triangular matrix, $|\mathbf{A}| = \prod_{i=1}^{n} a_{ii}$.

REVIEW: The table below summarizes equivalent conditions for the existence of an inverse matrix $\mathbf{A}^{-1}$ (where $\mathbf{A}$ has dimension $n \times n$).

$\mathbf{A}^{-1}$ exists	$\mathbf{A}^{-1}$ does not exist
$\mathbf{A}$ is nonsingular	$\mathbf{A}$ is singular
$\vert\mathbf{A}\vert \neq 0$	$\vert\mathbf{A}\vert = 0$
$\mathbf{A}$ has full rank	$\mathbf{A}$ has less than full rank
$r(\mathbf{A}) = n$	$r(\mathbf{A}) < n$
$\mathbf{A}$ has LIN rows (columns)	$\mathbf{A}$ does not have LIN rows (columns)
$\mathbf{Ax} = \mathbf{0}$ has one solution, $\mathbf{x} = \mathbf{0}$	$\mathbf{Ax} = \mathbf{0}$ has many solutions

EIGENVALUES: Suppose that $\mathbf{A}$ is a square matrix and consider the equations $\mathbf{Au} = \lambda\mathbf{u}$. Note that

$$\mathbf{Au} = \lambda\mathbf{u} \Longleftrightarrow \mathbf{Au} - \lambda\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}.$$

If $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{A} - \lambda\mathbf{I}$ must be singular (see last table). Thus, the values of λ which satisfy $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$ are those values where

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

This is called the **characteristic equation** of $\mathbf{A}$. If $\mathbf{A}$ is $n \times n$, then the characteristic equation is a polynomial (in λ) of degree n. The roots of this polynomial, say, $\lambda_1, \lambda_2, ..., \lambda_n$ are the **eigenvalues** of $\mathbf{A}$ (some of these may be zero or even imaginary). If $\mathbf{A}$ is a symmetric matrix, then $\lambda_1, \lambda_2, ..., \lambda_n$ must be real.

EIGENVECTORS: If $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues for $\mathbf{A}$, then vectors $\mathbf{u}_i$ satisfying

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i,$$

for $i = 1, 2, ..., n$, are called **eigenvectors**. Note that

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \Longrightarrow \mathbf{A}\mathbf{u}_i - \lambda_i\mathbf{u}_i = (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{u}_i = \mathbf{0}.$$

From our discussion on systems of equations and consistency, we know a general solution for $\mathbf{u}_i$ is given by $\mathbf{u}_i = [\mathbf{I} - (\mathbf{A} - \lambda_i\mathbf{I})^-(\mathbf{A} - \lambda_i\mathbf{I})]\mathbf{z}$, for $\mathbf{z} \in \mathcal{R}^n$.

Result MAR6.3. If λ_i and λ_j are eigenvalues of a symmetric matrix $\mathbf{A}$, and if $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors, $\mathbf{u}_i$ and $\mathbf{u}_j$, are orthogonal.
Proof. We know that $\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and $\mathbf{A}\mathbf{u}_j = \lambda_j\mathbf{u}_j$. The key is to recognize that

$$\lambda_i\mathbf{u}_i'\mathbf{u}_j = \mathbf{u}_i'\mathbf{A}\mathbf{u}_j = \lambda_j\mathbf{u}_i'\mathbf{u}_j,$$

which can only happen if $\lambda_i = \lambda_j$ or if $\mathbf{u}_i'\mathbf{u}_j = 0$. But $\lambda_i \neq \lambda_j$ by assumption. $\square$

PUNCHLINE: For a symmetric matrix $\mathbf{A}$, eigenvectors associated with distinct eigenvalues are orthogonal (we've just proven this) and, hence, are linearly independent. If the symmetric matrix $\mathbf{A}$ has an eigenvalue λ_k, of multiplicity m_k, then we can find m_k orthogonal eigenvectors of $\mathbf{A}$ which correspond to λ_k (Searle, pp 291). This leads to the following result (c.f., Christensen, pp 402):

Result MAR6.4. If $\mathbf{A}$ is a symmetric matrix, then there exists a basis for $\mathcal{C}(\mathbf{A})$ consisting of eigenvectors of nonzero eigenvalues. If λ is a nonzero eigenvalue of multiplicity m, then the basis will contain m eigenvectors for λ. Furthermore, $\mathcal{N}(\mathbf{A})$ consists of the eigenvectors associated with $\lambda = 0$ (along with $\mathbf{0}$).

SPECTRAL DECOMPOSITION: Suppose that $\mathbf{A}_{n\times n}$ is symmetric with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. The spectral decomposition of $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}'$, where

- $\mathbf{Q}$ is orthogonal; i.e., $\mathbf{Q}\mathbf{Q}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$,
- $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, a diagonal matrix consisting of the eigenvalues of $\mathbf{A}$; note that $r(\mathbf{D}) = r(\mathbf{A})$, because $\mathbf{Q}$ is orthogonal, and
- the columns of $\mathbf{Q}$ are orthonormal eigenvectors of $\mathbf{A}$.

Result MAR6.5. If $\mathbf{A}$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then

1. $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$
2. $tr(\mathbf{A}) = \sum_{i=1}^n \lambda_i$.

NOTE: These facts are also true for a general $n \times n$ matrix $\mathbf{A}$.

Proof (in the symmetric case). Write $\mathbf{A}$ in its Spectral Decomposition $\mathbf{A} = \mathbf{QDQ}'$. Note that $|\mathbf{A}| = |\mathbf{QDQ}'| = |\mathbf{DQ}'\mathbf{Q}| = |\mathbf{D}| = \prod_{i=1}^n \lambda_i$. Also, $tr(\mathbf{A}) = tr(\mathbf{QDQ}') = tr(\mathbf{DQ}'\mathbf{Q}) = tr(\mathbf{D}) = \sum_{i=1}^n \lambda_i$. □

Result MAR6.6. Suppose that $\mathbf{A}$ is symmetric. The rank of $\mathbf{A}$ equals the number of nonzero eigenvalues of $\mathbf{A}$.
Proof. Write $\mathbf{A}$ in its spectral decomposition $\mathbf{A} = \mathbf{QDQ}'$. Because $r(\mathbf{D}) = r(\mathbf{A})$ and because the only nonzero elements in $\mathbf{D}$ are the nonzero eigenvalues, the rank of $\mathbf{D}$ must be the number of nonzero eigenvalues of $\mathbf{A}$. □

Result MAR6.7. The eignenvalues of an idemptotent matrix $\mathbf{A}$ are equal to 0 or 1.
Proof. If λ is an eigenvalue of $\mathbf{A}$, then $\mathbf{Au} = \lambda\mathbf{u}$. Note that $\mathbf{A}^2\mathbf{u} = \mathbf{AAu} = \mathbf{A}\lambda\mathbf{u} = \lambda\mathbf{Au} = \lambda^2\mathbf{u}$. This shows that λ^2 is an eigenvalue of $\mathbf{A}^2 = \mathbf{A}$. Thus, we have $\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{Au} = \lambda^2\mathbf{u}$, which implies that $\lambda = 0$ or $\lambda = 1$. □

Result MAR6.8. If the $n \times n$ matrix $\mathbf{A}$ is idempotent, then $r(\mathbf{A}) = tr(\mathbf{A})$.
Proof. From the last result, we know that the eignenvalues of $\mathbf{A}$ are equal to 0 or 1. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ be a basis for $\mathcal{C}(\mathbf{A})$. Denote by $\mathcal{S}$ the subspace of all eigenvectors associated with $\lambda = 1$. Suppose $\mathbf{v} \in \mathcal{S}$. Then, because $\mathbf{Av} = \mathbf{v} \in \mathcal{C}(\mathbf{A})$, $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$. This means that any basis for $\mathcal{C}(\mathbf{A})$ is also a basis for $\mathcal{S}$. Furthermore, $\mathcal{N}(\mathbf{A})$ consists of eigenvectors associated with $\lambda = 0$ (because $\mathbf{Av} = 0\mathbf{v} = \mathbf{0}$). Thus,

$$\begin{aligned} n = \dim(\mathcal{R}^n) &= \dim[\mathcal{C}(\mathbf{A})] + \dim[\mathcal{N}(\mathbf{A})] \\ &= r + \dim[\mathcal{N}(\mathbf{A})], \end{aligned}$$

showing that $\dim[\mathcal{N}(\mathbf{A})] = n - r$. Since $\mathbf{A}$ has n eigenvalues, all are accounted for $\lambda = 1$ (with multiplicity r) and for $\lambda = 0$ (with multiplicity $n - r$). Now $tr(\mathbf{A}) = \sum_i \lambda_i = r$, the multiplicity of $\lambda = 1$. But $r(\mathbf{A}) = \dim[\mathcal{C}(\mathbf{A})] = r$ as well. □

TERMINOLOGY: Suppose that $\mathbf{x}$ is an $n \times 1$ vector. A **quadratic form** is a function $f : \mathcal{R}^n \to \mathcal{R}$ of the form

$$f(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j = \mathbf{x}'\mathbf{Ax}.$$

The matrix $\mathbf{A}$ is called the matrix of the quadratic form.

Result MAR6.9. If $\mathbf{x}'\mathbf{A}\mathbf{x}$ is any quadratic form, there exists a symmetric matrix $\mathbf{B}$ such that $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x}$.
Proof. Note that $\mathbf{x}'\mathbf{A}'\mathbf{x} = (\mathbf{x}'\mathbf{A}\mathbf{x})' = \mathbf{x}'\mathbf{A}\mathbf{x}$, since a quadratic form is a scalar. Thus,

$$\begin{aligned}\mathbf{x}'\mathbf{A}\mathbf{x} &= \frac{1}{2}\mathbf{x}'\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{A}'\mathbf{x} \\ &= \mathbf{x}'\left(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}'\right)\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x},\end{aligned}$$

where $\mathbf{B} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}'$. It is easy to show that $\mathbf{B}$ is symmetric. □

UPSHOT: In working with quadratic forms, we can, without loss of generality, assume that the matrix of the quadratic form is symmetric.

TERMINOLOGY: The quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be

- **nonnegative definite** (nnd) if $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$, for all $\mathbf{x} \in \mathcal{R}^n$.
- **positive definite** (pd) if $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$, for all $\mathbf{x} \neq \mathbf{0}$.
- **positive semidefinite** (psd) if $\mathbf{x}'\mathbf{A}\mathbf{x}$ is nnd but not pd.

TERMINOLOGY: A symmetric $n \times n$ matrix $\mathbf{A}$ is said to be nnd, pd, or psd if the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is nnd, pd, or psd, respectively.

Result MAR6.10. Let $\mathbf{A}$ be a symmetric matrix. Then

1. $\mathbf{A}$ pd $\Longrightarrow |\mathbf{A}| > 0$
2. $\mathbf{A}$ nnd $\Longrightarrow |\mathbf{A}| \geq 0$.

Result MAR6.11. Let $\mathbf{A}$ be a symmetric matrix. Then

1. $\mathbf{A}$ pd $\Longleftrightarrow$ all eigenvalues of $\mathbf{A}$ are positive
2. $\mathbf{A}$ nnd $\Longleftrightarrow$ all eigenvalues of $\mathbf{A}$ are nonnegative.

Result MAR6.12. A pd matrix is nonsingular. A psd matrix is singular. The converses are not true.

CONVENTION: If $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ matrices, we write $\mathbf{A}_1 \geq_{\text{nnd}} \mathbf{A}_2$ if $\mathbf{A}_1 - \mathbf{A}_2$ is nonnegative definite (nnd) and $\mathbf{A}_1 \geq_{\text{pd}} \mathbf{A}_2$ if $\mathbf{A}_1 - \mathbf{A}_2$ is positive definite (pd).

Result MAR6.13. Let $\mathbf{A}$ be an $m \times n$ matrix of rank r. Then $\mathbf{A}'\mathbf{A}$ is nnd with rank r. Furthermore, $\mathbf{A}'\mathbf{A}$ is pd if $r = n$ and is psd if $r < n$.
Proof. Let $\mathbf{x}$ be an $n \times 1$ vector. Then $\mathbf{x}'(\mathbf{A}'\mathbf{A})\mathbf{x} = (\mathbf{A}\mathbf{x})'\mathbf{A}\mathbf{x} \geq 0$, showing that $\mathbf{A}'\mathbf{A}$ is nnd. Also, $r(\mathbf{A}'\mathbf{A}) = r(\mathbf{A}) = r$. If $r = n$, then the columns of $\mathbf{A}$ are linearly independent and the only solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. This shows that $\mathbf{A}'\mathbf{A}$ is pd. If $r < n$, then

the columns of $\mathbf{A}$ are linearly dependent; i.e., there exists an $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$. Thus, $\mathbf{A}'\mathbf{A}$ is nnd but not pd, so it must be psd. $\square$

RESULT: A square matrix $\mathbf{A}$ is pd iff there exists a nonsingular lower triangular matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LL}'$. This is called the **Choleski Factorization** of $\mathbf{A}$. Monahan proves this result (see pp 258), provides an algorithm on how to find $\mathbf{L}$, and includes an example.

RESULT: Suppose that $\mathbf{A}$ is symmetric and pd. Writing $\mathbf{A}$ in its Spectral Decomposition, we have $\mathbf{A} = \mathbf{QDQ}'$. Because $\mathbf{A}$ is pd, $\lambda_1, \lambda_2, ..., \lambda_n$, the eigenvalues of $\mathbf{A}$, are positive. If we define $\mathbf{A}^{1/2} = \mathbf{QD}^{1/2}\mathbf{Q}'$, where $\mathbf{D}^{1/2} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})$, then $\mathbf{A}^{1/2}$ is symmetric and

$$\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{QD}^{1/2}\mathbf{Q}'\mathbf{QD}^{1/2}\mathbf{Q}' = \mathbf{QD}^{1/2}\mathbf{ID}^{1/2}\mathbf{Q}' = \mathbf{QDQ}' = \mathbf{A}.$$

The matrix $\mathbf{A}^{1/2}$ is called the **symmetric square root** of $\mathbf{A}$. See Monahan (pp 259-60) for an example.