1. For any matrix $\mathbf{A}$, prove that $\mathcal{R}(\mathbf{A}'\mathbf{A}) = \mathcal{R}(\mathbf{A})$.

2. Define the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Find $\mathcal{C}(\mathbf{A})$ and a basis for this space.
(b) Find $\mathcal{N}(\mathbf{A}')$ and a basis for this space.

3. In discussing the problem of calculating frequencies for different relatives, of pairs of genotypes at a single two-allele locus, one uses two matrices of conditional probabilities. They are

$$\mathbf{P} = \begin{bmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} p & q & 0 \\ \frac{1}{2}p & \frac{1}{2} & \frac{1}{2}q \\ 0 & p & q \end{bmatrix},$$

$\mathbf{P}$ for when the relatives have no genes identical by descent and $\mathbf{T}$ for when they have one gene identical by descent. With $p + q = 1$, $\mathbf{j}' = (1, 1, 1)$, and $\mathbf{S} = \frac{1}{4}\mathbf{I}_3 + \frac{1}{2}\mathbf{T} + \frac{1}{4}\mathbf{P}$, show that
(a) $\mathbf{Pj} = \mathbf{j}$
(b) $\mathbf{T}^2 = \frac{1}{2}(\mathbf{P} + \mathbf{T})$
(c) $\mathbf{P}^2 = \mathbf{P}$
(d) $\mathbf{T}^n = \mathbf{P} + \left(\frac{1}{2}\right)^{n-1}(\mathbf{T} - \mathbf{P})$, for all $n \geq 1$
(e) $\mathbf{S}^2 = \frac{1}{16}(\mathbf{I}_3 + 6\mathbf{T} + 9\mathbf{P})$.

4. Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix and $\mathbf{J}_n$ denote the $n \times n$ matrix, each of whose entries is 1. For $p \neq 0$ and $p + nq \neq 0$, show that

$$(p\mathbf{I}_n + q\mathbf{J}_n)^{-1} = p^{-1}\left(\mathbf{I}_n - \frac{q}{p + nq}\mathbf{J}_n\right).$$

5. Let $\mathbf{X}$ be an $n \times p$ matrix. Assuming that $(\mathbf{X}'\mathbf{X})^{-1}$ exists, define $\mathbf{M} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Show that $\mathbf{M}$ is a symmetric and that $\mathbf{M}^2 = \mathbf{M}$. In addition, show that $\mathbf{MX} = \mathbf{X}$ and that $\mathcal{C}(\mathbf{M}) = \mathcal{C}(\mathbf{X})$.

6. Suppose that $\mathcal{V}$ is vector space, and that $\mathcal{S}_1$ and $\mathcal{S}_2$ are both subspaces of $\mathcal{V}$. Define

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{v} \in \mathcal{V} : \mathbf{v} \in \mathcal{S}_1 \text{ and } \mathbf{v} \in \mathcal{S}_2\}.$$

Show that $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a subspace of $\mathcal{V}$.

7. Define the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Give a matrix whose column space contains $\mathcal{C}(\mathbf{A})$.
(b) Does $\mathbf{A}$ have linearly independent columns? Explain.
(c) Give a basis for the space spanned by the columns of $\mathbf{A}$.
(d) Find $\text{rank}(\mathbf{A})$.
(e) Are the columns of $\mathbf{A}$ mutually orthogonal? Explain.
(f) Find $\mathcal{N}(\mathbf{A}')$ and a basis for this space.

8. Prove that $\mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A})$ if and only if $\mathbf{AC} = \mathbf{B}$ for some matrix $\mathbf{C}$.

9. Suppose that $\mathcal{V}$ is a vector space and that $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m \in \mathcal{V}$.
(a) Prove that the set of all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m$; i.e.,

$$\mathcal{S} = \left\{ \mathbf{x} \in \mathcal{V} : \mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \right\}$$

is a subspace of $\mathcal{V}$.
(b) Does $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m\}$ form a basis for $\mathcal{S}$? If so, prove it. If not, provide a counterexample.

10. Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Find a basis for $\mathcal{C}(\mathbf{A})$.
(b) Find a vector $\mathbf{c} \neq \mathbf{0}$ such that $\mathbf{Ac} = \mathbf{0}$.
(c) Find a matrix $\mathbf{B}$ whose column space contains $\mathcal{C}(\mathbf{A})$.
(d) Find the orthogonal complement of $\mathcal{C}(\mathbf{A})$.

11. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be orthogonal complements in $\mathcal{R}^m$. Is it true that every vector in $\mathcal{R}^m$ is either in $\mathcal{S}_1$ or $\mathcal{S}_2$? If so, prove it. If not, give a counterexample.