1. Consider the GM model $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, where $\mathbf{X}$ is $N \times p$ with rank $r \le p$, $E(\mathbf{e}) = \mathbf{0}$ and $\text{cov}(\mathbf{e}) = \sigma^2\mathbf{I}$. Let $\widehat{\mathbf{b}}$ denote a least squares estimator of $\mathbf{b}$ and suppose that $\boldsymbol{\lambda}'\mathbf{b}$ is estimable.
(a) Assume $\mathbf{a}'\mathbf{y}$ is an unbiased estimator of $\boldsymbol{\lambda}'\mathbf{b}$. What the Gauss-Markov Theorem say about $\text{var}(\mathbf{a}'\mathbf{y})$ when compared to $\text{var}(\boldsymbol{\lambda}'\widehat{\mathbf{b}})$?
(b) Let $\widehat{\boldsymbol{\mu}} = \mathbf{X}\widehat{\mathbf{b}}$ and $\widetilde{\boldsymbol{\mu}} = \mathbf{A}'\mathbf{y}$ for a nonrandom $N \times N$ matrix $\mathbf{A}$. Assume that $E(\mathbf{A}'\mathbf{y}) = \mathbf{X}\mathbf{b}$. Show that $\text{cov}(\widetilde{\boldsymbol{\mu}}) - \text{cov}(\widehat{\boldsymbol{\mu}})$ is nnd.
(c) Let $\widetilde{\boldsymbol{\mu}} = \mathbf{A}'\mathbf{y}$ for a nonrandom $N \times N$ matrix $\mathbf{A}$. Assume that $E(\mathbf{A}'\mathbf{y}) = \mathbf{X}\mathbf{b}$. Show that

$$\sum_{i=1}^{N} \text{var}(\widetilde{\mu}_i) \ge r\sigma^2,$$

where $\widetilde{\mu}_i$ is the ith component of $\widetilde{\boldsymbol{\mu}}$.

2. Suppose that $\mathbf{V}$ is pd. Prove that $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$ is nnd.

3. Consider the linear model defined by

$$\begin{aligned} y_1 &= 2\theta + e_1 \\ y_2 &= \theta + e_2, \end{aligned}$$

where $e_1 = 2z_1 - z_2$ and $e_2 = z_1 + 2z_2$, and z_1 and z_2 are independent random variables with zero mean and constant variance σ^2.
(a) Write this model in $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ form. Find $E(\mathbf{y})$ and $\text{cov}(\mathbf{y})$.
(b) Compute the ordinary least squares (OLS) estimator of θ.
(c) Compute the generalized least squares (GLS) estimator of θ.
(d) Show that the OLS and GLS estimators are both unbiased.
(e) Compute the variance of both estimators and compare.

4. Suppose that $y_1, y_2, ..., y_N$ is an iid $\mathcal{U}(0, 2\theta)$ sample, where $\theta > 0$. Define $e_i = y_i - \theta$, for $i = 1, 2, ..., N$.
(a) Find the mean and covariance matrix of $\mathbf{e} = (e_1, e_2, ..., e_N)'$.
(b) Show that $\mathbf{y} = (y_1, y_2, ..., y_N)'$ follows a Gauss-Markov model.
(c) Find the BLUE of θ, say $\widehat{\theta}_{\text{OLS}}$. Give both a matrix expression and an expression in terms of simple summary statistics.
(d) Find c so that $\widehat{\theta} = cy_{(N)}$, where $y_{(N)} = \max\{y_1, y_2, ..., y_N\}$, is unbiased for θ and compute the variance of $\widehat{\theta}$.
(e) Compare the variances of $\widehat{\theta}_{\text{OLS}}$ and $\widehat{\theta}$. Are you surprised? Explain your findings in light of the Gauss-Markov Theorem.

5. Suppose that $y_1 \sim \text{beta}(\alpha, \beta)$, where $\alpha = \beta = 2$. Let $y_2 = 1 - y_1$, $y_3 = y_1 y_2$, and $\mathbf{y} = (y_1, y_2, y_3)'$. Compute $E(\mathbf{y}'\mathbf{U}\mathbf{y})$, where $\mathbf{U}$ is the matrix in Example 2.3 (notes, pp 61). **Use Result GM.3 to answer this question.**