1. Consider our general linear model $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$, where $E(\mathbf{e}) = \mathbf{0}$ and $\text{cov}(\mathbf{e}) = \sigma^2\mathbf{I}$. Suppose that $\mathbf{X}$ is $N \times p$ with rank $r < p$. Let $\widehat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}$ denote a least squares estimator of $\mathbf{b}$. Find $E(\widehat{\mathbf{b}})$ and $\text{cov}(\widehat{\mathbf{b}})$. Do your answers change when $r = p$?

2. Consider the regression model $y_i = \beta_0 + \beta_1 x_i + \beta_2(3x_i^2 - 2) + e_i$, for $i = 1, 2, 3$, where $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$.
(a) Put this model into $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$ form.
(b) Find the least-squares estimates of β_0, β_1, and β_2.
(c) Show that the least-squares estimates of β_0 and β_1 are unchanged if $\beta_2 = 0$. Why do you think this happens?

3. Consider an experiment to study the effect of baking time, x, on the breaking strength of a ceramic, y. The following eight data values were obtained:

x	2	6	8
y	15, 20, 25	21, 25, 29	33, 37

(a) Consider the cell means model $y_{ij} = \mu_i + e_{ij}$, for $i = 1, 2, 3$ and $j = 1, 2, ..., n_i$, where $n_1 = n_2 = 3$ and $n_3 = 2$. Put this model into $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$ form, where $\mathbf{b} = (\mu_1, \mu_2, \mu_3)'$.
(b) Consider the model $y_{ij} = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + e_{ij}$, where $x_1 = 2$, $x_2 = 6$, and $x_3 = 8$. Write this model as $\mathbf{y} = \mathbf{Wc} + \mathbf{e}$, where $\mathbf{c} = (\beta_0, \beta_1, \beta_2)'$.
(c) Show that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$. What does this imply about the fitted values and residuals for these two models? See the summary discussion on pp 50-51 (notes).

4. Let $\mathbf{P_X}$ be the perpendicular projection matrix onto $\mathcal{C}(\mathbf{X})$. Suppose that $\mathbf{a} \in \mathcal{C}(\mathbf{X})$. Show that $(\mathbf{P_X} - \mathbf{aa}')'(\mathbf{P_X} - \mathbf{aa}') = \mathbf{P_X} + (\mathbf{a}'\mathbf{a} - 2)\mathbf{aa}'$.

5. Consider the linear model $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$ with

$$\mathbf{y} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that $r(\mathbf{X}) = 3$. Find $\widehat{\mathbf{b}}_1$ and $\widehat{\mathbf{b}}_2$, two different solutions to the normal equations $\mathbf{X}'\mathbf{Xb} = \mathbf{X}'\mathbf{y}$. With your solutions, show that $\mathbf{X}\widehat{\mathbf{b}}_1 = \mathbf{X}\widehat{\mathbf{b}}_2$. Also show directly that $\mathbf{y} - \mathbf{X}\widehat{\mathbf{b}}_1 = \mathbf{y} - \mathbf{X}\widehat{\mathbf{b}}_2 \in \mathcal{N}(\mathbf{X}')$.