1. For any matrix $\mathbf{A}$, show that $\mathcal{R}(\mathbf{A}'\mathbf{A}) = \mathcal{R}(\mathbf{A})$.

2. Define the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Find $\mathcal{C}(\mathbf{A})$ and a basis for this space.
(b) Find $\mathcal{N}(\mathbf{A}')$ and a basis for this space.
(c) Find 2 different generalized inverses of $\mathbf{A}$, say $\mathbf{A}_1^-$ and $\mathbf{A}_2^-$. Check your work by showing that $\mathbf{A}\mathbf{A}_1^-\mathbf{A} = \mathbf{A}$ and $\mathbf{A}\mathbf{A}_2^-\mathbf{A} = \mathbf{A}$.
(d) Compute $\mathbf{M}_\mathbf{A}$, the perpendicular projection matrix onto $\mathcal{C}(\mathbf{A})$.
(e) Compute $\mathbf{I} - \mathbf{M}_\mathbf{A}$, the perpendicular projection matrix onto $\mathcal{N}(\mathbf{A}')$.

3. Let $\mathbf{A}_{n\times p}$, $\mathbf{b}_{p\times 1}$, $\mathbf{c}_{n\times 1}$, and suppose that the equations $\mathbf{Ab} = \mathbf{c}$ are consistent. Let $\mathbf{x}_{n\times 1}$, $\mathbf{u}_{p\times 1}$, and $\mathbf{X}_{p\times n}$. Let $\mathbf{A}_1^-$ and $\mathbf{A}_2^-$ be two generalized inverses of $\mathbf{A}$. Let $\mathbf{I}$ denote the $n \times n$ identity matrix.
(a) Let $\mathbf{b}^*$ be a solution to $\mathbf{Ab} = \boldsymbol{c}$. Show that $\mathbf{b}^* + \mathbf{uc}'\{(\mathbf{A}_1^-)'\mathbf{A}' - \mathbf{I}\}\mathbf{x}$ is also a solution.
(b) Show that $\mathbf{A}_1^- + \mathbf{X}(\mathbf{A}\mathbf{A}_2^- - \mathbf{I})$ is a generalized inverse of $\mathbf{A}$.

4. (a) Suppose that $\mathbf{A}$ is an $n \times n$ symmetric matrix. Prove that $\mathbf{A}$ is idempotent if and only if $r(\mathbf{A}) + r(\mathbf{I} - \mathbf{A}) = n$.
(b) Suppose that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$ are all idempotent. Prove that $\mathbf{AB} = \mathbf{0}$ and $\mathbf{BA} = \mathbf{0}$.

5. Let $\mathbf{P}$ be an $n\times n$ orthogonal matrix and let $\mathbf{A}$ be an $n\times n$ symmetric and idempotent matrix. Define $\mathbf{D} = \mathbf{P}'\mathbf{AP}$.
(a) Show that $\mathbf{D}$ is a perpendicular projection matrix.
(b) Show that if $\mathbf{A}$ is nonnegative definite, then so is $\mathbf{D}$.