DIRECTIONS: There are 4 questions. Point totals are in []. This exam is worth 50 points. Show all of your work. Explain all of your reasoning. This exam is closed-book, closed-notes. No calculators are allowed.

1. [10] State the Gauss-Markov Theorem.

2. [18] Let $\mathbf{y}$ be a 6×1 vector,

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix},$$

and $\mathbf{b} = (\mu, \theta_1, \theta_2, \theta_3, \theta_4)'$. Assume the linear model $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ holds, where $E(\mathbf{e}) = \mathbf{0}$ and $\mathbf{X}$ and $\mathbf{b}$ are nonrandom. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ denote the columns of $\mathbf{X}$. Note that $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 = \mathbf{0}$.
(a) Find the components of $E(\mathbf{y})$ in terms of $\mu, \theta_1, \theta_2, \theta_3$, and θ_4.
(b) Show that $\mathbf{x}_0$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ are linearly independent.
(c) Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)'$. Give conditions on $\boldsymbol{\lambda}$, of the form $\boldsymbol{\lambda}'\mathbf{c}_i = 0$, $i = 1, 2, ..., s$, that are necessary and sufficient for $\boldsymbol{\lambda}'\mathbf{b}$ to be estimable. What is the value of s for this model?
(d) Show that $\mu + \theta_3 - \theta_4$ is estimable.
(e) Give a nonestimable function of the form $\boldsymbol{\lambda}'\mathbf{b}$. Explain your answer.
(f) Explain briefly how you could use your answer in part (e) to force a particular solution to the normal equations $\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}$. If you did this, would your nonestimable $\boldsymbol{\lambda}'\mathbf{b}$ in part (e) become estimable? Explain.

3. [10] Suppose that $\mathbf{y} = \mathbf{X}\mathbf{b}+\mathbf{e}$, where $\mathbf{X}$ is $N \times p$ with rank $r = p$ and $\mathbf{e} \sim \mathcal{N}_N(\mathbf{0}, \sigma^2\mathbf{V})$. Let $\widehat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ denote the ordinary least squares (OLS) estimator of $\mathbf{b}$. Find $E(\widehat{\mathbf{b}})$ and $\text{cov}(\widehat{\mathbf{b}})$.

4. [12] Suppose that $\mathbf{y} = (y_1, y_2)' \sim \mathcal{N}_2(\boldsymbol{\mu}, \mathbf{V})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Also, take

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) Find the distribution of $\mathbf{b}'\mathbf{y}$.
(b) Find the distribution of $\mathbf{A}\mathbf{y} + \mathbf{b}$.
(c) Compute $E(\mathbf{y}'\mathbf{A}\mathbf{y})$.