DIRECTIONS: There are 4 questions. Each question is worth 10 points. Show all of your work. Explain all of your reasoning. This part is closed-book, closed-notes. No calculators are allowed.

1. A researcher studies the effects of 3 different diets on weight gains in rats. She chooses 12 rats at random (from a large population) and assigns the rats to exactly one of the 3 diets. She also records the covariate x (initial weight), because she knows the response y (weight gain) will depend on x. Assume that 4 rats are assigned to each diet.
(a) Write out a linear model, in non-matrix notation, that relates the response y to the covariate x and the diets. Clearly define all of your notation. Use appropriate subscripts to denote the different diets and rats.
(b) Take your model in (a) and write it in the form $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$. Define all vectors and matrices.
(c) What assumptions must be true for your model to be a Guass-Markov model?

2. Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Find a basis for $\mathcal{C}(\mathbf{A})$.
(b) Find a vector $\mathbf{c} \neq \mathbf{0}$ such that $\mathbf{Ac} = \mathbf{0}$.
(c) Find a generalized inverse of $\mathbf{A}'\mathbf{A}$.
(d) If $\mathbf{y}$ is random vector with mean $\boldsymbol{\mu} = (1, -1, 0)'$ and covariance matrix $\mathbf{I}_3$, find $E(\mathbf{Ay})$ and $\text{cov}(\mathbf{Ay})$.

3. Suppose the system $\mathbf{Ax} = \mathbf{c}$ is consistent and that $\mathbf{G}$ is a generalized inverse of $\mathbf{A}$.
(a) What is a particular solution to the system? the general solution?
(b) If $\mathbf{A}$ is symmetric, prove that $\frac{1}{2}(\mathbf{G} + \mathbf{G}')$ is a generalized inverse of $\mathbf{A}$.
(c) Prove that the generalized inverse in (b) is symmetric. This shows that there does exist a generalized inverse of $\mathbf{A}$, $\mathbf{A}$ symmetric, that is symmetric itself.

4. True or False. A true statement is one that is always true. A false statement may be true some of the time, but not always. Each question is worth 2 points. No partial credit will be given, so no explanation is necessary.
(a) True or False. If the columns of $\mathbf{A}$ are linearly dependent, then $|\mathbf{A}| = 0$.
(b) True or False. If $\mathbf{A}$ is idempotent, then $r(\mathbf{A}) = tr(\mathbf{A})$.
(c) True or False. A linear system $\mathbf{Ax} = \mathbf{c}$ is consistent if $\mathbf{c} \in \mathcal{C}(\mathbf{A})$.
(d) True or False. For the general linear model $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$, the normal equations have a unique solution when $\mathbf{X}$ is of full column rank.
(e) True or False. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are random vectors. If $\mathbf{z} = \mathbf{x} + \mathbf{y}$, then $\text{cov}(\mathbf{z}) = \text{cov}(\mathbf{x}) + \text{cov}(\mathbf{y})$.