1. Do the following problems in Chapter 2 from Cryer and Chan: 2.5, 2.7, 2.10, 2.11, 2.12, 2.19, and 2.27. Although not always stated, it is understood that $\{e_t\}$ is a zero mean white noise process with $\text{var}(e_t) = \sigma_e^2$.

2.5 Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k.

(a) Find the mean function for $\{Y_t\}$.

$$\begin{aligned} E(Y_t) &= E(5 + 2t + X_t) \\ &= E(5) + E(2t) + E(X_t) \\ &= 5 + 2t + 0 \\ &= 5 + 2t \end{aligned}$$

(b) Find the autocovariance function for $\{Y_t\}$.

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(5 + 2t + X_t, 5 + 2[t-k] + X_{t-k}) \\ &= \text{Cov}(X_t, X_{t-k}) \\ &= \gamma_k \end{aligned}$$

(c) Is $\{Y_t\}$ stationary? Why or why not?

This process is not stationary because its mean function $\mu_t = E(Y_t)$ increases with time.

2.7 Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and autocovariance function for $\{W_t\}$.

In order to show that $\{W_t\}$ is stationary, we must show that its mean and autocovariance functions are free of t. Since $\{Y_t\}$ is stationary, we know that its mean and autocovariance functions are free of t. Therefore, we will denote the mean function by the constant μ and the autocovariance by γ_k.

$$\begin{aligned} E(W_t) &= E(Y_t - Y_{t-1}) \\ &= E(Y_t) - E(Y_{t-1}) \\ &= \mu - \mu \\ &= 0 \end{aligned}$$

$$\begin{aligned}
\text{Cov}(W_t, W_{t-k}) &= \text{Cov}(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\
&= \text{Cov}(Y_t, Y_{t-k}) - \text{Cov}(Y_t, Y_{t-k-1}) - \text{Cov}(Y_{t-1}, Y_{t-k}) + \text{Cov}(Y_{t-1}, Y_{t-k-1}) \\
&= \text{Cov}(Y_t, Y_{t-k}) - \text{Cov}(Y_t, Y_{t-[k+1]}) - \text{Cov}(Y_{t-1}, Y_{t-k}) + \text{Cov}(Y_{t-1}, Y_{t-[k+1]}) \\
&= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\
&= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}
\end{aligned}$$

Since the mean and autocovariance functions do not depend on t, $\{W_t\}$ is stationary. Therefore, we just showed that the difference of a stationary series is also stationary.

(b) Show that $U_t = \nabla^2 Y_t = \nabla[Y_t - Y_{t-1}] = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary. (You need not find the mean and autocovariance functions for $\{U_t\}$.)

Expanding upon Part (a), if the difference of a stationary series is stationary, then every subsequent difference must also be stationary. Therefore, this part follows directly from part (a).

2.10 Let $\{X_t\}$ be a zero-mean, unit-variance stationary process with autocorrelation function ρ_k. Suppose that μ_t is a nonconstant function and that σ_t is a positive-valued nonconstant function. The observed series is formed as $Y_t = \mu_t + \sigma_t X_t$.

(a) Find the mean and covariance function for the $\{Y_t\}$ process.

$$\begin{aligned}
E(Y_t) &= E(\mu_t + \sigma_t X_t) \\
&= E(\mu_t) + E(\sigma_t X_t) \\
&= \mu_t + \sigma_t(0) \\
&= \mu_t
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k}) \\
&= \sigma_t \sigma_{t-k} \text{Cov}(X_t, X_{t-k}) \\
&= \sigma_t \sigma_{t-k} \text{Corr}(X_t, X_{t-k}) \sqrt{\text{Var}(X_t)\text{Var}(X_{t-k})} \\
&= \sigma_t \sigma_{t-k} \rho_k \sqrt{(1)(1)} \\
&= \sigma_t \sigma_{t-k} \rho_k
\end{aligned}$$

(b) Show that the autocorrelation function for the $\{Y_t\}$ process depends only on the time lag. Is the $\{Y_t\}$ process stationary?

$$\begin{aligned}
\mathrm{Corr}(Y_t, Y_{t-k}) &= \frac{\mathrm{Cov}(Y_t, Y_{t-k})}{\sqrt{\mathrm{Var}(Y_t)\mathrm{Var}(Y_{t-k})}} \\
&= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\sigma_t^2 \sigma_{t-k}^2}} \\
&= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} \\
&= \rho_k
\end{aligned}$$

Even though the autocorrelation function for $\{Y_t\}$ is constant over time, its mean function is not. Therefore, this process is not stationary.

(c) Is it possible to have a time series with a constant mean and with $\mathrm{Corr}(Y_t, Y_{t-k})$ free of t but with $\{Y_t\}$ not stationary?

Parts (a) and (b) showed that a series can have an autocovariance function that depends on t, yet still have an autocorrelation function that is free of t. Therefore, yes, this is possible.

2.11 Suppose $\mathrm{Cov}(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E(X_t) = 3t$.

(a) Is $\{X_t\}$ stationary?

Since the mean function for $\{X_t\}$ involves t, the series is not stationary.

(b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned}
E(Y_t) &= E(7 - 3t + X_t) \\
&= E(7) - E(3t) + E(X_t) \\
&= 7 - 3t + 3t \\
&= 7
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(7 - 3t + X_t, 7 - 3t + X_{t-k}) \\
&= \mathrm{Cov}(X_t, X_{t-k}) \\
&= \gamma_k
\end{aligned}$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary.

2.12 Suppose that $Y_t = e_t - e_{t-12}$. Show that $\{Y_t\}$ is stationary and that, for $k > 0$, its autocorrelation function is nonzero only for lag $k = 12$.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned} E(Y_t) &= E(e_t - e_{t-12}) \\ &= E(e_t) - E(e_{t-12}) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(e_t - e_{t-12}, e_{t-k} - e_{t-k-12}) \\ &= \mathrm{Cov}(e_t, e_{t-k}) - \mathrm{Cov}(e_t, e_{t-k-12}) - \mathrm{Cov}(e_{t-12}, e_{t-k}) + \mathrm{Cov}(e_{t-12}, e_{t-k-12}) \\ &= \mathrm{Cov}(e_t, e_{t-k}) - 0 - \mathrm{Cov}(e_{t-12}, e_{t-k}) + \mathrm{Cov}(e_{t-12}, e_{t-k-12}) \\ &= -\mathrm{Cov}(e_{t-12}, e_{t-k}) \\ &= \left\{ \begin{array}{ll} \mathrm{Var}(e_t) + \mathrm{Var}(e_{t-12}), & k = 0 \\ -\mathrm{Var}(e_{t-12}), & k = 12 \\ 0, & k \neq 0, 12 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 2\sigma_e^2, & k = 0 \\ -\sigma_e^2, & k = 12 \\ 0, & k \neq 0, 12 \end{array} \right\} \end{aligned}$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary.

2.19 Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t\}$ is called a **random walk with drift**.

(a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \ldots + e_1$.

This is a proof by induction.

Base Case: Show $t = 1$

Plugging $t = 1$ into the equation, it obviously yields $Y_1 = 1\theta_0 + e_1 = \theta_0 + e_1$. This proves the base case.

Inductive Case: Assume $Y_t = t\theta_0 + e_t + e_{t-1} + \ldots + e_1$. Show $Y_{t+1} = (t+1)\theta_0 + e_{t+1} + e_t + e_{t-1} + \ldots + e_1$.

$$\begin{aligned} Y_{t+1} &= \theta_0 + Y_t + e_{t+1} \\ &= \theta_0 + \underbrace{t\theta_0 + e_t + e_{t-1} + \ldots + e_1}_{=Y_t} + e_{t+1} \\ &= (t+1)\theta_0 + e_{t+1} + e_t + e_{t-1} + \ldots + e_1 \end{aligned}$$

This proves the inductive case. This completes the proof.

(b) Find the mean function for $\{Y_t\}$.

$$\begin{aligned}E(Y_t) &= E(t\theta_0 + e_t + e_{t-1} + \ldots + e_1)\\ &= E(t\theta_0) + E(e_t) + E(e_{t-1}) + \ldots + E(e_1)\\ &= t\theta_0 + 0 + 0 + \ldots + 0\\ &= t\theta_0\end{aligned}$$

(c) Find the autocovariance function for $\{Y_t\}$.

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(t\theta_0 + e_t + e_{t-1} + \ldots + e_1, (t-k)\theta_0 + e_{t-k} + e_{t-k-1} + \ldots + e_1)\\ &= \text{Cov}(e_t + e_{t-1} + \ldots + e_1, e_{t-k} + e_{t-k-1} + \ldots + e_1)\\ &= \text{Cov}(e_{t-k}, e_{t-k}) + \text{Cov}(e_{t-k-1}, e_{t-k-1}) + \ldots + \text{Cov}(e_1, e_1)\\ &= \text{Var}(e_{t-k}) + \text{Var}(e_{t-k-1}) + \ldots + \text{Var}(e_1)\\ &= \sigma_e^2 + \sigma_e^2 + \ldots + \sigma_e^2\\ &= (t-k)\sigma_e^2\end{aligned}$$

Note that this process is not stationary because its mean and autocovariance functions depend on time.

2.27 For a fixed, positive integer r and constant ϕ, consider the time series defined by $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^r e_{t-r}$.

(a) Show that this process is stationary for any value of ϕ.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned}E(Y_t) &= E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^r e_{t-r})\\ &= E(e_t) + E(\phi e_{t-1}) + E(\phi^2 e_{t-2}) + \ldots + E(\phi^r e_{t-r})\\ &= 0 + \phi(0) + \phi^2(0) + \ldots + \phi^r(0)\\ &= 0\end{aligned}$$

$\text{Cov}(Y_t, Y_{t-k}) =$

$$\text{Cov}(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^r e_{t-r}, e_{t-k} + \phi e_{t-k-1} + \phi^2 e_{t-k-2} + \ldots + \phi^r e_{t-k-r})$$

Case 1: $r < k$.

$$e_t + \phi e_{t-(1)} + \ldots + \phi^r e_{t-(r)}$$

$$e_{t-k} + \phi e_{t-k-(1)} + \ldots + \phi^r e_{t-k-(r)}$$

If $r < k$, then none of the white noise subscripts match. Therefore, the covariance is zero.

Case 2: $r = k$.

$$\begin{aligned}&e_t + \phi e_{t-(1)} + \ldots + \phi^r e_{t-(r)}\\ &\qquad\qquad\qquad\qquad e_{t-k} \;+ \phi e_{t-k-(1)} + \ldots + \phi^r e_{t-k-(r)}\end{aligned}$$

The only white noise term that matches is e_{t-r}. Therefore,

$$\text{Cov}(Y_t, Y_{t-k}) = \text{Cov}(\phi^r e_{t-r}, e_{t-r}) = \phi^k \text{Var}(e_{t-r}) = \phi^k \sigma_e^2.$$

Case 3: $r > k$.

$$\begin{aligned}&e_t + \phi e_{t-(1)} + \ldots + \phi^k e_{t-(k)} + \phi^{k+1} e_{t-(k+1)} + \ldots + \phi^r \quad e_{t-(r)}\\ &\qquad\qquad\qquad\qquad e_{t-k} \quad + \phi \quad e_{t-k-(1)} + \ldots + \phi^{r-k} e_{t-k-(r-k)} + \ldots + \phi^r e_{t-k-(r)}\end{aligned}$$

The subscripts match at all e_s's between e_{t-k} and e_{t-r} (inclusive). Therefore,

$$\begin{aligned}&\text{Cov}(Y_t, Y_{t-k}) =\\ &= \text{Cov}(\phi^k e_{t-k}, e_{t-k}) + \text{Cov}(\phi^{k+1} e_{t-k-1}, \phi e_{t-k-1}) + \ldots + \text{Cov}(\phi^r e_{t-r}, \phi^{r-k} e_{t-r})\\ &= \phi^k \text{Var}(e_{t-k}) + \phi^{k+2} \text{Var}(e_{t-k-1}) + \ldots + \phi^{2r-k} \text{Var}(e_{t-r})\\ &= \sigma_e^2 (\phi^k + \phi^{k+2} + \ldots + \phi^{2r-k})\\ &= \sigma_e^2 \phi^k (1 + \phi^2 + \ldots + \phi^{2r-2k})\\ &= \sigma_e^2 \phi^k \left(\frac{1 - \phi^{2r-2k+2}}{1 - \phi^2} \right)\end{aligned}$$

The last step following from summing a finite geometric series. Finally, we have

$$\text{Cov}(Y_t, Y_{t-k}) = \left\{ \begin{array}{ll} 0, & r < k \\ \phi^k \sigma_e^2, & r = k \\ \sigma_e^2 \phi^k \left(\dfrac{1 - \phi^{2(r-k+1)}}{1 - \phi^2} \right), & r > k \end{array} \right\}$$

Since neither of these functions depends on t, we know that $\{Y_t\}$ is stationary.

(b) Find the autocorrelation function.

First, we must note that, by stationarity,

$$\begin{aligned}
\text{Var}(Y_t) &= \text{Var}(Y_{t-k}) \\
&= \gamma_0 \\
&= \sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2} \right)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\text{Corr}(Y_t, Y_{t-k}) &= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} \\
&= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right) \sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right)}} \\
&= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right)} \\
&= \left\{ \begin{array}{ll}
\dfrac{0}{\sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right)}, & r < k \\
\dfrac{\phi^k \sigma_e^2}{\sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right)}, & r = k \\
\dfrac{\sigma_e^2 \phi^k \left(\frac{1-\phi^{2(r-k+1)}}{1-\phi^2}\right)}{\sigma_e^2 \left(\frac{1-\phi^{2(r+1)}}{1-\phi^2}\right)}, & r > k
\end{array} \right\} \\
&= \left\{ \begin{array}{ll}
0, & r < k \\
\dfrac{\phi^r(1-\phi^2)}{1-\phi^{2(r+1)}}, & r = k \\
\dfrac{\phi^k(1-\phi^{2(r-k+1)})}{1-\phi^{2(r+1)}}, & r > k
\end{array} \right\}
\end{aligned}$$

2. Suppose that Z_1 and Z_2 are uncorrelated random variables with zero mean and unit variance. Consider the process defined by

$$Y_t = Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t$$

where $e_t \sim \text{iid } \mathcal{N}(0, \sigma_e^2)$ and $\{e_t\}$ is independent of both Z_1 and Z_2.
(a) Prove that $\{Y_t\}$ is stationary.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$
\begin{aligned}
E(Y_t) &= E[Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t] \\
&= E[Z_1 \cos(\omega t)] + E[Z_2 \sin(\omega t)] + E[e_t] \\
&= \cos(\omega t) E[Z_1] + \sin(\omega t) E[Z_2] + E[e_t] \\
&= \cos(\omega t)(0) + \sin(\omega t)(0) + 0 \\
&= 0
\end{aligned}
$$

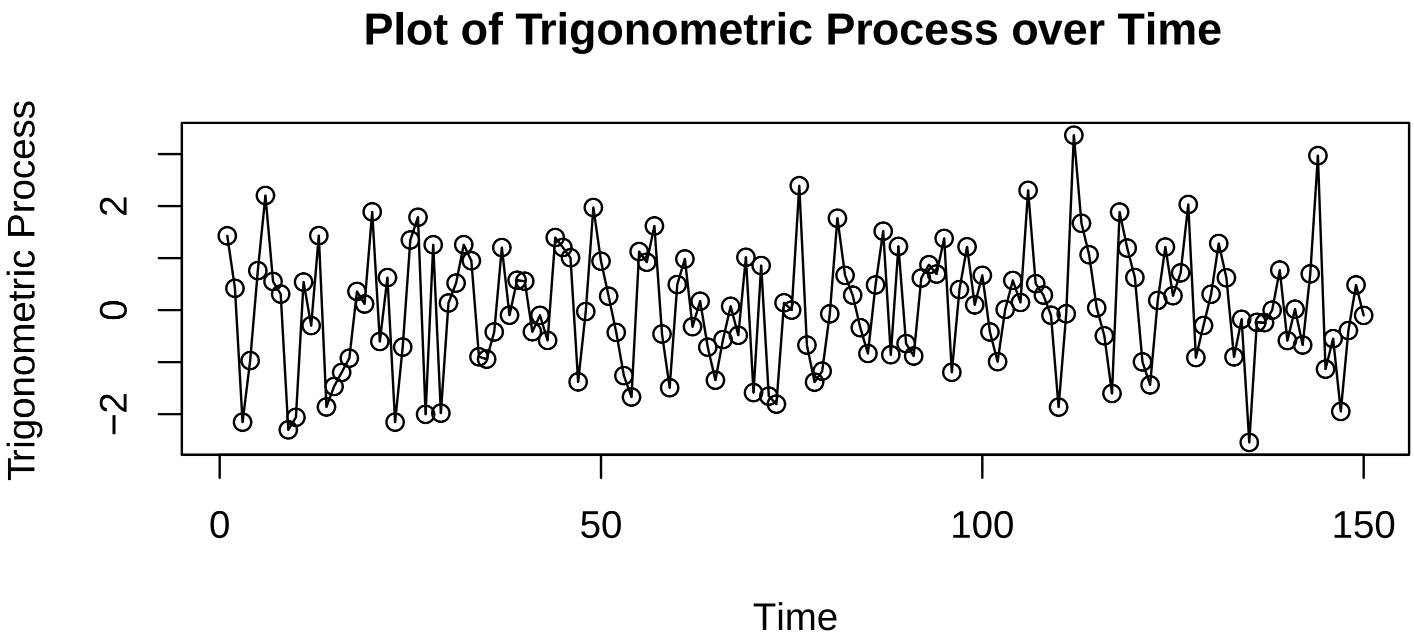
$$
\begin{aligned}
&\text{Cov}(Y_t, Y_{t-k}) = \\
&\quad \text{Cov}(Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t, Z_1 \cos(\omega[t-k]) + Z_2 \sin(\omega[t-k]) + e_{t-k}) \\
&= \cos(\omega t) \cos(\omega[t-k]) \text{Var}(Z_1) + \sin(\omega t) \sin(\omega[t-k]) \text{Var}(Z_2) + \text{Cov}(e_t, e_{t-k}) \\
&= \cos(\omega t) \cos(\omega[t-k])(1) + \sin(\omega t) \sin(\omega[t-k])(1) + 0 \\
&= \cos(\omega t) \cos(\omega[t-k]) + \sin(\omega t) \sin(\omega[t-k]) \\
&= \cos(\omega t - \omega[t-k]) \\
&\text{Trigonometric Identity}: \ \{\cos(\text{x}) \cos(\text{y}) + \sin(\text{x}) \sin(\text{y}) = \cos(\text{x} - \text{y})\} \\
&= \cos(\omega k)
\end{aligned}
$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary.

(b) Let Z_1 and Z_2 be independent $\mathcal{N}(0, 1)$ random variables, and set $\sigma_e^2 = 1$ and $\omega = 0.5$. Use the following R commands to simulate $n = 150$ observations from the $\{Y_t\}$ process:

```
> omega = 1
> Z = rnorm(2,0,1)
> e.t = rnorm(150,0,1)
> Y.t = e.t*0
> for (i in 1:length(e.t)){
      Y.t[i] = Z[1]*cos(omega*i) + Z[2]*sin(omega*i) + e.t[i]}
> plot(Y.t,ylab="Trigonometric process",xlab="Time",type="o")
```

Describe the appearance of your time series.



First, we should note that the problem stated that $\omega = .5$. However, in the R code, we used $\omega = 1$. If you used $\omega = .5$ in your code, then the frequency of your process would be lower, making it much easier to recognize the sinusoidal pattern in the process. Zooming in quite far on this series, you can make out a definite sinusoidal pattern with a very high frequency. In fact, we should expect this since it is a trigonometric process. There does appear to be a decent amount of variability evident in the magnitude of the peaks and troughs. It is also important to note that this process seems to have a constant mean process at zero. There are two factors that could cause this plot to look quite unusual. First, if Z_1 and Z_2 are both close to zero, then the trigonometric trend in the graph would be drowned out by the random error. Second, if Z_1 and Z_2 are very close to one another, then at places where $\cos(\omega t)$ = -$\sin(\omega t)$, then Y_t would be mostly white noise.

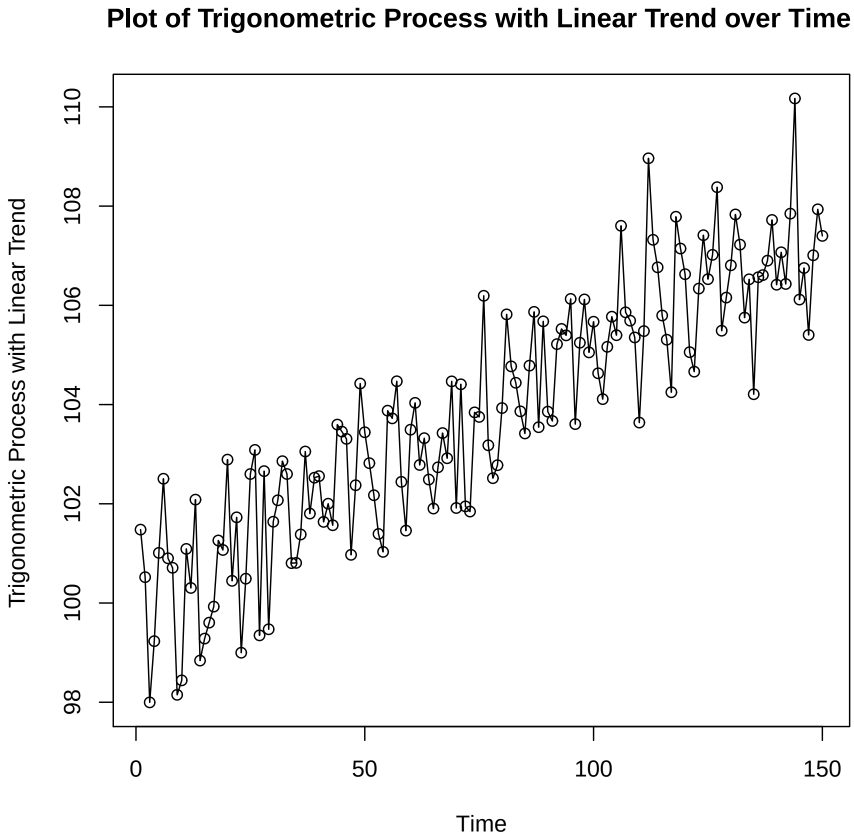
(c) Amend the R code above to simulate a realization of the process

$$\widetilde{Y}_t = \beta_0 + \beta_1 t + Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t,$$

where $\beta_0 = 100$, $\beta_1 = 0.05$, $\sigma_e^2 = 1$ and $\omega = 0.5$. To do this, replace the last three lines of the R code above with

```
> Y.tilde = e.t*0
> for (i in 1:length(e.t)){
   Y.tilde[i] = 100 + 0.05*i + Z[1]*cos(omega*i) + Z[2]*sin(omega*i) + e.t[i]}
> plot(Y.tilde,ylab="Trig process with linear trend",xlab="Time",type="o")
```

Does your $\{\widetilde{Y}_t\}$ process appear to be stationary? What is the effect of adding the linear trend term $\beta_0 + \beta_1 t$ to the model?

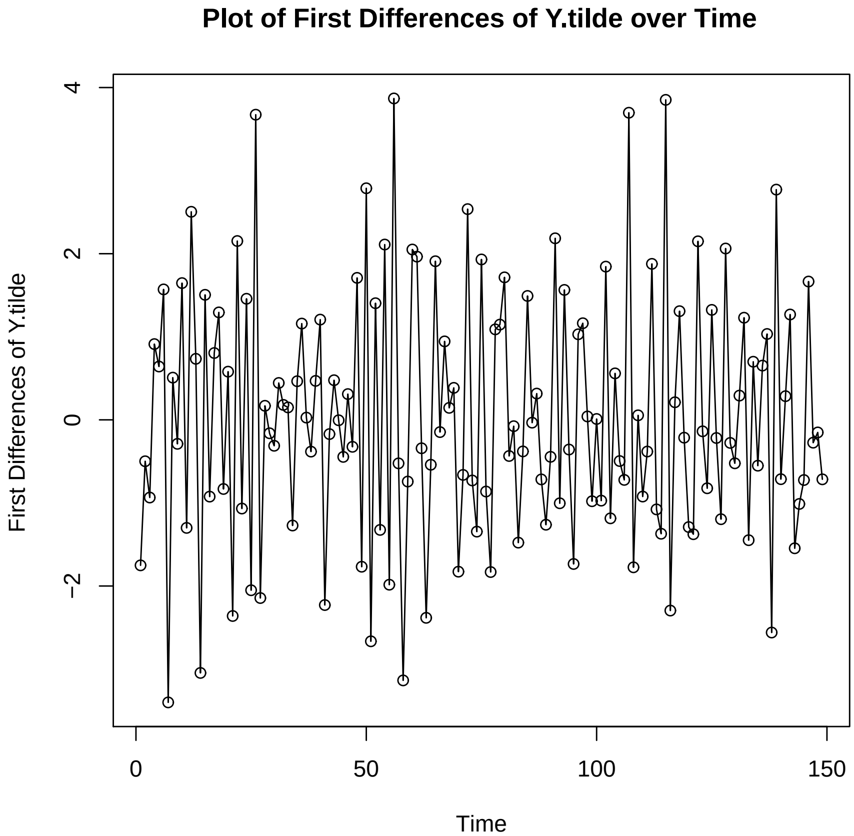


This process is definitely no longer stationary. Adding the linear trend caused the process to linearly increase with time.

(d) Plot the first differences of your simulated $\{\widetilde{Y}_t\}$ process. To do this, use

```
> diff.Y.tilde = diff(Y.tilde)
> plot(diff.Y.tilde,ylab="First differences of Y.tilde",xlab="Time",type="o")
```

Describe the appearance of this first difference process $\{\nabla\widetilde{Y}_t\}$. In particular, does it appear to be stationary in the mean level? Are you surprised? Discuss the behavior of each process: $\{Y_t\}$, $\{\widetilde{Y}_t\}$, and $\{\nabla\widetilde{Y}_t\}$, and how they relate to each other.



This series appears very similar to the plot of $\{Y_t\}$, at least in the mean level. It appears to be stationary. In fact, differencing a series reduces its polynomial degree by one. Therefore, the linear trend became flat after differencing.

(e) In part (c), suppose that instead of the added linear trend $\beta_0 + \beta_1 t$, the added trend was quadratic, say, $\beta_0 + \beta_1 t + \beta_2 t^2$. Do you think the first differences of the quadratic trend version of the process in part (c) would be stationary? Explain your intuition.

As we just stated, differencing a series reduces its polynomial trend degree by one. Therefore, differencing a series with a quadratic trend would yield a series with a linear trend. This is intuitive if you understand the base for differentiation in calculus, which behaves similarly to differencing in this regard.

3. Give an example of a process $\{Y_t\}$ that satisfies the following. *In each part, prove that your answer is correct.*

(i) A process with constant mean but variance that increases with time.

Let Y_t be independent $N(0, t)$ random variables. This series has a constant mean of zero and a variance that increases linearly with time.

(ii) A stationary process whose autocovariance does not go to zero as time lag goes to infinity.

Let $Y_t = e_0$, for $t = 1, 2, \ldots$, where $e_0 \sim N(0, 1)$.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned} E(Y_t) &= E(e_0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(e_0, e_0) \\ &= \text{Var}(e_0) \\ &= 1 \end{aligned}$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary. Also, the autocovariance function of $\{Y_t\}$ does not approach zero, it remains constant for all lags k.

(iii) A nonstationary process whose autocovariance depends only on time lag.

Let $Y_t = t + e_t$.

In order to show that this process is nonstationary, let's show that its mean function depends on t.

$$\begin{aligned} E(Y_t) &= E(t + e_t) \\ &= E(t) + E(e_t) \\ &= t + 0 \\ &= t \end{aligned}$$

Now, let's show that its autocovariance function depends only on lag k.

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(t + e_t, t - k + e_{t-k}) \\ &= \text{Cov}(e_t, e_{t-k}) \\ &= \left\{ \begin{array}{ll} \text{Var}(e_t), & k = 0 \\ 0 & k > 0 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \sigma_e^2, & k = 0 \\ 0 & k > 0 \end{array} \right\} \end{aligned}$$

(iv) A stationary process that has nonzero autocorrelation only at lag $k = 1$.

Let $Y_t = e_t + e_{t-1}$.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned} E(Y_t) &= E(e_t + e_{t-1}) \\ &= E(e_t) + E(e_{t-1}) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(e_t + e_{t-1}, e_{t-k} + e_{t-k-1}) \\ &= \text{Cov}(e_t, e_{t-k}) + \text{Cov}(e_t, e_{t-k-1}) + \text{Cov}(e_{t-1}, e_{t-k}) + \text{Cov}(e_{t-1}, e_{t-k-1}) \\ &= \left\{ \begin{array}{ll} \text{Var}(e_t) + \text{Var}(e_{t-1}), & k = 0 \\ \text{Var}(e_{t-1}), & k = 1 \\ 0, & k > 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \sigma_e^2 + \sigma_e^2, & k = 0 \\ \sigma_e^2, & k = 1 \\ 0, & k > 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 2\sigma_e^2, & k = 0 \\ \sigma_e^2, & k = 1 \\ 0, & k > 1 \end{array} \right\} \end{aligned}$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary. Now, let's find the autocorrelation function for $\{Y_t\}$.

$$\begin{aligned} \text{Corr}(Y_t, Y_{t-k}) &= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} \\ &= \left\{ \begin{array}{ll} 1, & k = 0 \\ \frac{\sigma_e^2}{\sqrt{2\sigma_e^2(2\sigma_e^2)}}, & k = 1 \\ \frac{0}{\sqrt{2\sigma_e^2(2\sigma_e^2)}}, & k > 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 1, & k = 0 \\ \frac{1}{2}, & k = 1 \\ 0, & k > 1 \end{array} \right\} \end{aligned}$$

As you can see, the only non-trivial correlation is at lag $k = 1$.

(v) A stationary process that has nonzero autocorrelation only at lag $k = 4$.

Let $Y_t = e_t + e_{t-4}$.

In order for $\{Y_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$
\begin{aligned}
E(Y_t) &= E(e_t + e_{t-4}) \\
&= E(e_t) + E(e_{t-4}) \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(e_t + e_{t-4}, e_{t-k} + e_{t-k-4}) \\
&= \text{Cov}(e_t, e_{t-k}) + \text{Cov}(e_t, e_{t-k-4}) + \text{Cov}(e_{t-4}, e_{t-k}) + \text{Cov}(e_{t-4}, e_{t-k-4}) \\
&= \left\{ \begin{array}{lr} \text{Var}(e_t) + \text{Var}(e_{t-4}), & k = 0 \\ \text{Var}(e_{t-4}), & k = 4 \\ 0, & \text{o.w.} \end{array} \right\} \\
&= \left\{ \begin{array}{lr} \sigma_e^2 + \sigma_e^2, & k = 0 \\ \sigma_e^2, & k = 4 \\ 0, & \text{o.w.} \end{array} \right\} \\
&= \left\{ \begin{array}{lr} 2\sigma_e^2, & k = 0 \\ \sigma_e^2, & k = 4 \\ 0, & \text{o.w.} \end{array} \right\}
\end{aligned}
$$

Since neither of these functions depend on t, $\{Y_t\}$ is stationary. Now, let's find the autocorrelation function for $\{Y_t\}$.

$$
\begin{aligned}
\text{Corr}(Y_t, Y_{t-k}) &= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} \\
&= \left\{ \begin{array}{lr} 1, & k = 0 \\ \frac{\sigma_e^2}{\sqrt{2\sigma_e^2(2\sigma_e^2)}}, & k = 4 \\ \frac{0}{\sqrt{2\sigma_e^2(2\sigma_e^2)}}, & \text{o.w.} \end{array} \right\} \\
&= \left\{ \begin{array}{lr} 1, & k = 0 \\ \frac{1}{2}, & k = 4 \\ 0, & \text{o.w.} \end{array} \right\}
\end{aligned}
$$

As you can see, the only non-trivial correlation is at lag $k = 4$.

(vi) A nonstationary process whose first differences are stationary.

Let $Y_t = t$.

In order to show that this process is nonstationary, let's show that its mean function depends on t.

$$
\begin{aligned}
E(Y_t) &= E(t) \\
&= t
\end{aligned}
$$

Let $W_t = \nabla Y_t$.

In order for $\{W_t\}$ to be stationary, we must show that its mean and autocovariance functions do not depend on t.

$$\begin{aligned}
E(W_t) &= E(Y_t - Y_{t-1}) \\
&= E(Y_t) - E(Y_{t-1}) \\
&= t - (t-1) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(W_t, W_{t-1}) &= \text{Cov}(Y_t - Y_{t-1}, Y_{t-1} - Y_{t-2}) \\
&= \text{Cov}(t - [t-1], [t-1] - [t-2]) \\
&= 0
\end{aligned}$$

Since neither of these functions depend on t, $\{W_t\}$ is stationary.