1. Let T denote the failure time (in years) for a population of individuals, and suppose that the hazard function, for $t > 0$, is given by

$$\lambda_T(t) = \alpha e^{\beta t},$$

for $\alpha > 0$ and $\beta > 0$. In each of the following parts, your final answer will be a function of α and β.
(a) Derive the survival distribution $S_T(t)$.
(b) Compute the median survival.
(c) What is the median survival among individuals known to be alive at 3 years?
(d) If $\beta < 0$, this would imply that the hazard rate would tend to zero as time t becomes larger. This may then imply that a certain proportion of individuals would never die (i.e., they are cured). Compute the proportion of individuals that do not die if $\beta < 0$.

2. Let T be a lifetime random variable with pdf $f_T(t)$ and survivor function $S_T(t)$. Assume that $E(T) < \infty$ so that $tS_T(t) \to 0$ as $t \to \infty$.
(a) Show that the pdf of T, conditional on $T > t_0$ for $t_0 \geq 0$, is given by

$$f(t|T > t_0) = \begin{cases} \dfrac{f_T(t)}{S_T(t_0)}, & t > t_0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Define $h(t_0) = E(T - t_0|T > t_0)$. Show that

$$h(t_0) = \int_{t_0}^{\infty} \frac{S_T(t)}{S_T(t_0)} dt.$$

The value $h(t_0)$ gives your mean remaining lifetime, given that you have survived up through time t_0. This is also called the **mean residual lifetime**.
(c) Now suppose that $f_T(t) = t\exp(-t)$, for $t > 0$. Show that for $t_0 > 0$,

$$S_T(t_0) = (t_0 + 1)e^{-t_0} \quad \text{and} \quad h(t_0) = \frac{t_0 + 2}{t_0 + 1}.$$

3. *A mixed exponential model.* Suppose that a population consists of individuals for which lifetimes T are exponentially distributed, but that the hazard function λ varies across individuals (this a Bayesian/hierarchical feel to it). To be specific, suppose that the distribution of T, given λ, has pdf

$$f_{T|\lambda}(t|\lambda) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

and that λ itself varies according to a gamma distribution with pdf

$$g(\lambda) = \begin{cases} \dfrac{\lambda^{k-1} e^{-\lambda/\alpha}}{\alpha^k \Gamma(k)}, & \lambda > 0 \\ 0, & \text{otherwise.} \end{cases}$$

where $\alpha > 0$ and $k > 0$ are known.

(a) Find the unconditional pdf and survivor function of T. That is, find $f_T(t)$ and $S_T(t)$.
(b) Show that the unconditional hazard function of T is

$$h_T(t) = \frac{k\alpha}{1+\alpha t}.$$

(c) Show that $h_T(t)$ is monotone decreasing. For which types of populations might this hazard function be appropriate?

4. The following table shows data on the time to diagnosis of AIDS among 143 HIV infected intravenous drug users followed over an eight-year period.

Year since entry into study	Number under observation at beginning of interval	Number diagnosed with AIDS during interval	Number censored or withdrawn
$[0,1)$	143	2	3
$[1,2)$	138	4	2
$[2,3)$	132	4	4
$[3,4)$	124	11	10
$[4,5)$	103	18	4
$[5,6)$	81	19	7
$[6,7)$	55	21	9
$[7,8)$	25	18	7

(a) For $t = 1, 2, ..., 8$, find the life-table estimate of the survival function $S_T(t)$, where T denotes the time until diagnosis of AIDS. Denote this estimate by $\widehat{S}_T(t)$.
(b) Plot your estimate $\widehat{S}_T(t)$ versus time.
(c) Find an estimate of $\text{var}[\widehat{S}_T(t)]$ at time $t = 1, 2, ..., 8$.
(d) Find a 95 percent confidence interval for $S_T(5)$, the true proportion of individuals not diagnosed with AIDS at time $t = 5$.

5. A survival study involves subjects with inoperative lung cancer, and the endpoint T is time until death (measured in days). The data that follow are death/censoring times for $n = 25$ patients. Values with a "+" are censored observations (patients that have not died at the time of analysis).

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139, 304, 193, 248, 27, 210, 134, 203, 320+, 50, 306+, 22, 290+, 276+,
136, 231+, 217, 196, 186, 144, 68, 131+, 121+, 108, 99
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(a) Use R to construct the Kaplan-Meier estimate of the true survival distribution $S_T(t)$. Also attach a 95 percent confidence band for $S_T(t)$.
(b) Report a point estimate for the median survival time $\phi_{0.5}$. Also, report a 95 percent confidence interval for $\phi_{0.5}$.