GROUND RULES:

- This exam contains 5 questions; each question is worth 10 points. The maximum number of points on this exam is 50.
- This is a closed-book and closed-notes exam.
- You may use a calculator if you wish, but SHOW ALL OF YOUR WORK AND EXPLAIN ALL OF YOUR REASONING!!!
- A standard normal table is attached on the last page.
- *Any discussion or otherwise inappropriate communication between examinees, as well as the appearance of any unnecessary material, will be dealt with severely.*
- Print your name **at the top of this page in the upper right hand corner**.
- You have 80 minutes to complete this exam. GOOD LUCK!

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. The lifetime of a certain brand of industrial light bulb, Y, is assumed to follow a **gamma** distribution with shape parameter $\alpha = 3$ and scale parameter $\beta = 6$. A random sample (i.e., an iid sample) of $n = 25$ light bulbs will be taken; denote these measurements by $Y_1, Y_2, ..., Y_{25}$. Approximate the probability that the sample mean $\overline{Y}$ will be larger than 21.

2. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from an **exponential** distribution with mean θ. Recall that the exponential(θ) probability density function (pdf) is given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta}e^{-y/\theta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

and that the cumulative distribution function (cdf) is given by

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y/\theta}, & y > 0. \end{cases}$$

Consider the **two estimators**

$$\widehat{\theta}_1 = nY_{(1)} \qquad \widehat{\theta}_2 = \overline{Y},$$

where $Y_{(1)}$ is the minimum of $Y_1, Y_2, ..., Y_n$, and $\overline{Y}$ is the sample mean. Recall that the pdf of the minimum order statistic $Y_{(1)}$, where positive, is given by

$$f_{Y_{(1)}}(y) = nf_Y(y)[1 - F_Y(y)]^{n-1}.$$

(a) Show that $Y_{(1)}$ follows an exponential distribution with mean θ/n.
(b) Show that both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unbiased estimators of θ.
(c) Compute the variance of both estimators. Which estimator would you prefer?

3. Suppose that $Z_1, Z_2, ..., Z_6$ is an iid sample from a $\mathcal{N}(0, 1)$ distribution. Suppose that $Z_7 \sim \mathcal{N}(0, 1)$ as well and that Z_7 is independent of $Z_1, Z_2, ..., Z_6$. **For each of the following parts, provide sufficient justification (in words) as to why your answer is correct.**

Find the distribution of

(a) $\overline{Z} = \frac{1}{6}\sum_{i=1}^{6} Z_i$

(b) $T = \sum_{i=1}^{6}(Z_i - \overline{Z})^2$

(c) $U = \sqrt{3}Z_7/\sqrt{Z_1^2 + Z_2^2 + Z_3^2}$

(d) $V = (Z_1^2 + Z_2^2 + Z_3^2)/(Z_4^2 + Z_5^2 + Z_6^2)$.

4. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from a **Poisson** distribution with mean θ.
(a) Give **two** unbiased estimators of θ.
(b) Find an unbiased estimator of $\tau(\theta) = \theta^2$.

5. Suppose that we have **two independent samples**:

$$Y_{11}, Y_{12}, ..., Y_{1n} \sim \text{iid } \mathcal{N}(\mu_1, \sigma^2)$$
$$Y_{21}, Y_{22}, ..., Y_{2n} \sim \text{iid } \mathcal{N}(\mu_2, \sigma^2).$$

Notice that the two population variances are the same and that the sample sizes are equal. The population means μ_1 and μ_2 are unknown parameters. The common population variance σ^2 is also an unknown parameter. Define

$$\begin{aligned}
\overline{Y}_{1+} = \frac{1}{n}\sum_{j=1}^{n} Y_{1j} &= \text{sample mean for sample 1} \\
\overline{Y}_{2+} = \frac{1}{n}\sum_{j=1}^{n} Y_{2j} &= \text{sample mean for sample 2} \\
S_1^2 = \frac{1}{n-1}\sum_{j=1}^{n}(Y_{1j} - \overline{Y}_{1+})^2 &= \text{sample variance for sample 1} \\
S_2^2 = \frac{1}{n-1}\sum_{j=1}^{n}(Y_{2j} - \overline{Y}_{2+})^2 &= \text{sample variance for sample 2.}
\end{aligned}$$

(a) Find the distribution of $U = \overline{Y}_{1+} - \overline{Y}_{2+}$.
(b) Find the distribution of

$$W = \frac{(n-1)S_1^2}{\sigma^2} + \frac{(n-1)S_2^2}{\sigma^2} = \frac{(n-1)S_1^2 + (n-1)S_2^2}{\sigma^2}.$$

(c) Explain why U and W are independent.
(d) If $\mu_1 - \mu_2 = 0$, find a statistic, which is a function of U and W, that has an F distribution. What are its degrees of freedom? Remember that a statistic can not depend on unknown parameters.