

Here is a summary of some of the exact distributional results that we have discovered. Each of these can be proved using the distribution function technique, the method of transformations, or using an mgf argument.

1.  $Y \sim \mathcal{N}(0, 1) \implies U = Y^2 \sim \chi^2(1)$ .
2.  $Y \sim \mathcal{U}(0, 1) \implies U = -\ln Y \sim \text{exponential}(1)$ . A generalization of this is that  $Y \sim \mathcal{U}(0, 1) \implies U = -\theta \ln Y \sim \text{exponential}(\theta)$ .
3.  $Y \sim \text{exponential}(\alpha) \implies U = Y^{1/m} \sim \text{Weibull}(m, \alpha)$ . Conversely,  $Y \sim \text{Weibull}(m, \alpha) \implies U = Y^m \sim \text{exponential}(\alpha)$ .
4.  $Y \sim \text{beta}(\alpha, \beta) \implies U = 1 - Y \sim \text{beta}(\beta, \alpha)$ .
5.  $Y \sim \text{gamma}(\alpha, \beta) \implies U = 2Y/\beta \sim \chi^2(2\alpha)$ .
6.  $Y_1, Y_2, \dots, Y_n \sim \text{iid Bernoulli}(p) \implies U = \sum_i Y_i \sim b(n, p)$ .
7.  $Y_1, Y_2, \dots, Y_n \sim \text{independent gamma}(\alpha_i, \beta) \implies U = \sum_i Y_i \sim \text{gamma}(\sum_i \alpha_i, \beta)$ .

- A special case of this arises when  $\alpha_i = 1$  for each  $i$ . The  $\text{gamma}(1, \beta)$  and  $\text{exponential}(\beta)$  models are the same model. So, the sum of iid  $\text{exponential}(\beta)$  random variables has a  $\text{gamma}(n, \beta)$  distribution.
- Another very important application of this arises if  $\alpha_i = \nu_i/2$ , where  $\nu_i$  is an integer, and  $\beta = 2$ . In this case,  $Y_1, Y_2, \dots, Y_n \sim \text{independent } \chi^2(\nu_i)$ , and  $U = \sum_i Y_i \sim \chi^2(\sum_i \nu_i)$ ; i.e., the degrees of freedom add.
- Combining (5) and (7): If  $Y_1, Y_2, \dots, Y_n$  is an iid sample of  $\text{exponential}(\beta)$  observations, then

$$\frac{2 \sum_i Y_i}{\beta} \sim \chi^2(2n).$$

8.  $Y_1, Y_2, \dots, Y_n \sim \text{independent Poisson}(\lambda_i) \implies U = \sum_i Y_i \sim \text{Poisson}(\sum_i \lambda_i)$ .
9. If  $Y \sim \mathcal{N}(\mu, \sigma^2)$ , then

$$Z = \frac{Y - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

10.  $Y_1, Y_2, \dots, Y_n \sim$  independent  $\mathcal{N}(\mu_i, \sigma_i^2) \implies U = \sum_i a_i Y_i \sim \mathcal{N}(\sum_i a_i \mu_i, \sum_i a_i^2 \sigma_i^2)$ .

- A special case arises when  $Y_1, Y_2, \dots, Y_n \sim$  iid  $\mathcal{N}(\mu, \sigma^2)$ . In this case,  $U = \sum_i a_i Y_i \sim \mathcal{N}(\mu \sum_i a_i, \sigma^2 \sum_i a_i^2)$ .

- A special case of this iid result arises when  $a_i = 1/n$  for each  $i$ . Then,  $U = \sum_i a_i Y_i \equiv \bar{Y} \sim \mathcal{N}(\mu, \sigma^2/n)$ .

- Another special case of this iid result arises when  $a_i = 1$  for each  $i$ . Then,  $U = \sum_i Y_i \sim \mathcal{N}(n\mu, n\sigma^2)$ .

11.  $Y_1, Y_2, \dots, Y_n \sim$  independent  $\mathcal{N}(\mu_i, \sigma_i^2)$ . Then,

$$Z_i = \frac{Y_i - \mu_i}{\sigma_i} \sim \mathcal{N}(0, 1)$$

and  $U = \sum_i Z_i^2 \sim \chi^2(n)$ , since  $Z_1^2, Z_2^2, \dots, Z_n^2$  are iid  $\chi^2(1)$ .

12.  $Y_1, Y_2, \dots, Y_n \sim$  iid geometric( $p$ )  $\implies U = \sum_i Y_i \sim$  nib( $n, p$ ).

13.  $Y_1, Y_2 \sim$  iid  $\mathcal{N}(0, 1) \implies U = Y_1/Y_2 \sim$  Cauchy.

14.  $Y \sim \mathcal{U}(-\pi/2, \pi/2) \implies U = \tan Y \sim$  Cauchy.

15.  $Y_1, Y_2, \dots, Y_n \sim$  iid  $\mathcal{N}(\mu, \sigma^2) \implies (n-1)S^2/\sigma^2 \sim \chi^2(n-1)$ .

16. If  $Z \sim \mathcal{N}(0, 1)$ ,  $W \sim \chi^2(\nu)$ , and  $Z$  and  $W$  are independent, then

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t(\nu).$$

17. If  $W_1 \sim \chi^2(\nu_1)$ ,  $W_2 \sim \chi^2(\nu_2)$ , and  $W_1$  and  $W_2$  are independent, then

$$F = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F(\nu_1, \nu_2).$$

18. If  $T \sim t(\nu)$ , then  $T^2 \sim F(1, \nu)$ .

19. If  $F \sim F(\nu_1, \nu_2)$ , then  $1/F \sim F(\nu_2, \nu_1)$ .

20. If  $W \sim F(\nu_1, \nu_2)$ , then  $(\nu_1/\nu_2)W/[1 + (\nu_1/\nu_2)W] \sim$  beta( $\nu_1/2, \nu_2/2$ ).

21. If  $Y \sim$  beta( $\theta, 1$ ), then  $U = -\ln Y \sim$  exponential( $1/\theta$ ). Also, if  $Y \sim$  beta( $1, \theta$ ), then  $U = -\ln(1 - Y) \sim$  exponential( $1/\theta$ ).