1. I wrote this solution on Thursday, September 6, before the US Open Women Semifinals were played (they are over now). Still alive in the draw were J. Henin (#1), V. Williams (#12), S. Kuznetsova (#4), and A. Chakvetadze (#6). Consider the experiment of observing the US Open 2007 Women's Champion. There are 4 outcomes in S; namely,

$$S = \{H, W, K, C\},$$

where H denotes Henin, W denotes Williams, etc. These sample points are probably not equally likely. For example, Henin is the top-ranked player in the world, Chakvetadze has never been in a Grand Slam final (such as the US Open), Williams has won 6 slams, and Kuznetsova has won the US Open before (2006) but that is her only slam title. Note that in the semifinals, Henin will play Williams in the top half of the bracket; Kuznetsova will play Chakvetadze in the bottom half.

Based on the seedings and past performances, I will use the following probability model, based on my subjective opinion:

$$\begin{aligned} P(H) &= 0.40 \\ P(W) &= 0.25 \\ P(K) &= 0.25 \\ P(C) &= 0.10. \end{aligned}$$

Here are some events that we may be interested in (there are others):

$$\begin{aligned} A_1 &= \{\text{winner comes from top half of bracket}\} \\ A_2 &= \{\text{winner comes from bottom half of bracket}\} \\ A_3 &= \{\text{winner is Russian}\} \\ A_4 &= \{\text{winner is African-American}\}. \end{aligned}$$

Note that Kuznetsova and Chakvetadze are Russian. Williams is the only African-American. Under my assumed model ("assumed" is the key word here):

$$\begin{aligned} P(A_1) &= 0.40 + 0.25 = 0.65. \\ P(A_2) &= 0.25 + 0.10 = 0.35. \\ P(A_3) &= 0.25 + 0.10 = 0.35. \\ P(A_4) &= 0.25. \end{aligned}$$

2. (a) Let $a_1 < a_2 < \cdots < a_r$ be an enumeration of the set of integers $1, 2, ..., n$ and let $b_1 < b_2 < \cdots < b_s$ be an enumeration of the set of integers $1, 2, ..., m$. The sample space S consists of sample points E_i of the form

$$E_i = (W_{a_1}, W_{a_2}, ..., W_{a_r}, R_{b_1}, R_{b_2}, ..., R_{b_s}),$$

where $W_{a_1}, W_{a_2}, ..., W_{a_r}$ correspond to the r white balls chosen and $R_{b_1}, R_{b_2}, ..., R_{b_s}$ correspond to the s red balls chosen.

(b) Assuming that each point in S is equally likely, each sample point in S has probability $1/N$, where

$$N = \binom{n}{r}\binom{m}{s},$$

obtained using the multiplication rule; i.e.,

$$\binom{n}{r} = \#\text{ ways to choose } r \text{ white balls from } n$$
$$\binom{m}{s} = \#\text{ ways to choose } s \text{ white balls from } m.$$

(c) When $r = 5$, $n = 55$, $s = 1$, and $m = 42$, we have, for each sample point E_i,

$$P(E_i) = \frac{1}{\binom{55}{5}\binom{42}{1}} = \frac{1}{146107962} \approx 0.000000006844.$$

This is why I do not play Powerball.

3. We need to show that for this sample space S and for this probability measure P, the following properties hold:

(1) $P(A) \geq 0$, for every $A \subseteq S$

(2) $P(S) = 1$

(3) If $A_1, A_2, ...$ is a countable sequence of pairwise mutually exclusive events (i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$) in S, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

To verify (1), note that over the interval $S = (0, 1)$, the function $f(y) = 6y(1 - y)$ is strictly positive (graph it!). Because integrating a positive function can not give a negative result (area), the first property holds; that is,

$$P(A) = \int_A f(y)dy \geq 0.$$

It is interesting to note that $P(A) = 0$ when A consists of a finite or countable collection of singletons (here, a singleton is a particular number between 0 and 1). Also, $P(A) > 0$ when A is an interval or a countable union of intervals (some but not all of which could be singletons). To verify (2), note that

$$P(S) = \int_S 6y(1-y)dy = \int_0^1 6y(1-y)dy = \int_0^1 (6y - 6y^2)dy = \left[3y^2 - 2y^3\Big|_0^1\right] = 3 - 2 = 1.$$

To verify (3), suppose that $A_1, A_2, ...$ is a countable sequence of pairwise disjoint events. Because the A_i's are disjoint, the area under $f(y)$ over the sets $A_1, A_2, ...$ is simply the sum of the individual areas; that is,

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \int_{\bigcup_{i=1}^{\infty} A_i} f(y)dy \\ &= \underbrace{\int_{A_1} f(y)dy + \int_{A_2} f(y)dy + \int_{A_3} f(y)dy + \cdots}_{\text{sum of the individual areas}} \\ &= \sum_{i=1}^{\infty} \int_{A_i} f(y)dy = \sum_{i=1}^{\infty} P(A_i). \end{aligned}$$

This is simply the countable additivity property of Riemann integrals. Thus, the Kolmogorov axioms are satisfied.

4. The sample space here consists of all 7-tuples of the form

$$E_i = (a_1, a_2, a_3, a_4, a_5, a_6, a_7),$$

where $a_j \in \{1, 2, 3, 4, 5, 6, 7\}$ denotes the numbered day on which accident j occurs; $j = 1, 2, ..., 7$. For example, the point $(3, 4, 4, 3, 7, 1, 2)$ represents that the seven accidents occurred on days 3, 4, 4, 3, 7, 1, and 2, respectively. Note that there are $N = 7^7$ different sample points in S. This follows from the multiplication rule for counting.

(a) Note that A is satisfied if any of the $n_A = 7$ sample points are observed: $(1, 1, 1, 1, 1, 1, 1)$, $(2, 2, 2, 2, 2, 2, 2)$, ..., $(7, 7, 7, 7, 7, 7, 7)$. Thus,

$$P(A) = \frac{7}{7^7} \approx 0.0000085,$$

assuming that each outcome in S is equally likely.

(b) We need to count n_B the number of ways that B can occur, that is, we need to count those sample points E_i where the a_j entries are all different; for example, a sample point like $(6, 3, 4, 1, 7, 2, 5)$. How many such sample points are there? The answer is the number of ways to permute the integers $1, 2, 3, 4, 5, 6, 7$, or $n_B = 7!$. Thus,

$$P(A) = \frac{7!}{7^7} \approx 0.00612,$$

assuming that each outcome in S is equally likely.

5. Define the events

$$\begin{aligned} A &= \{\text{customer insures more than one car}\} \\ B &= \{\text{customer insures sports car}\}. \end{aligned}$$

We are given the following information: $P(A) = 0.70$, $P(B) = 0.20$, and $P(B|A) = 0.15$.
(a) In this part, we want to compute $P(\overline{A} \cap \overline{B})$. By DeMorgan's Law, we have

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B),$$

where the last step follows from the complement rule. We now need to compute $P(A \cup B)$. From Inclusion-Exclusion, we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.70 + 0.20 - P(A \cap B). \end{aligned}$$

We now need to compute $P(A \cap B)$! From the Multiplication Law of Probability, we have

$$P(A \cap B) = P(B|A)P(A) = 0.15(0.70) = 0.105.$$

Thus,

$$P(A \cup B) = 0.70 + 0.20 - 0.105 = 0.795.$$

Finally,

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.795 = 0.205.$$

(b) In this part, we want $P(A|B)$. From the definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.105}{0.20} = 0.525.$$

6. (a) *Proof.* Suppose that $P(A|B) = P(A|\overline{B})$. It suffices to show that $P(A) = P(A|B)$. By LOTP, we have that

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|\overline{B})P(\overline{B}) \\ &= P(A|B)P(B) + P(A|B)P(\overline{B}) \\ &= P(A|B)[P(B) + P(\overline{B})] \\ &= P(A|B), \end{aligned}$$

since $P(B) + P(\overline{B}) = 1$.
(b) *Proof.* Suppose that both $P(A|C) > P(B|C)$ and $P(A|\overline{C}) > P(B|\overline{C})$. Again, by LOTP, we can write $P(A)$ as

$$\begin{aligned} P(A) &= P(A|C)P(C) + P(A|\overline{C})P(\overline{C}) \\ &> P(B|C)P(C) + P(B|\overline{C})P(\overline{C}) \\ &= P(B), \end{aligned}$$

since $P(B|C)P(C) + P(B|\overline{C})P(\overline{C}) = P(B)$ by LOTP. The penultimate step is true by assumption.
(c) *Proof.* If A, B, and C are disjoint, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$, by Axiom 3, and the statement holds. Otherwise, we can use Inclusion-Exclusion to write

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\quad - P(B \cap C) + P(A \cap B \cap C) \\ &\leq P(A) + P(B) + P(C) - P(A \cap B) + P(A \cap B \cap C). \end{aligned}$$

This last statement is true because $P(A \cap C) \geq 0$ and $P(B \cap C) \geq 0$. Now note that $(A \cap B \cap C) \subseteq (A \cap B)$; thus, $P(A \cap B \cap C) \leq P(A \cap B)$, by monotonicity, and $P(A \cap B \cap C) - P(A \cap B) \leq 0$. This means that

$$\begin{aligned} P(A \cup B \cup C) &\leq P(A) + P(B) + P(C) \underbrace{-P(A \cap B) + P(A \cap B \cap C)}_{\leq 0} \\ &\leq P(A) + P(B) + P(C), \end{aligned}$$

which establishes the result.

7. Define the events

$$\begin{aligned} A &= \{+ \text{ sent by transmitter}\} \\ B &= \{+ \text{ received from Relay 2}\}. \end{aligned}$$

We are given that $P(A) = 0.6$. We want to compute $P(A|B)$. From Bayes Rule, we know that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}.$$

Thus, we need to compute $P(B|A)$ and $P(B|\overline{A})$. Note that $P(\overline{A}) = 1 - P(A) = 0.4$.

To compute $P(B|A)$, we realise that there are 2 distinct ways that a + signal can be received by the receiver, given that a + signal originated from the transmitter.

- The first way is that a + signal is sent from Relay 1 (i.e., the signal is not reversed at Relay 1) and a + signal is received from Relay 2 (i.e., Relay 2 does not reverse the signal either); this occurs with probability $0.75 \times 0.75 = (0.75)^2$, because the relays are assumed to be independent.
- The second way is that a − signal is sent from Relay 1 (i.e., the + signal sent by the transmitter is reversed at Relay 1) and a + signal is received from Relay 2 (i.e., the − signal from Relay 1 is reversed at Relay 2); this occurs with probability $0.25 \times 0.25 = (0.25)^2$, because the relays are assumed to be independent.

Thus,

$$P(B|A) = (0.75)^2 + (0.25)^2 = 0.625.$$

To compute $P(B|\overline{A})$, we realise that there are 2 distinct ways that a + signal can be received by the receiver, given that a − signal originated from the transmitter.

- The first way is that a + signal is sent from Relay 1 (i.e., the signal is reversed at Relay 1) and a + signal is received from Relay 2 (i.e., Relay 2 does not reverse the signal); this occurs with probability $0.25 \times 0.75 = 0.1875$, because the relays are assumed to be independent.

- The second way is that a $-$ signal is sent from Relay 1 (i.e., the $-$ signal sent by the transmitter is not reversed at Relay 1) and a $+$ signal is received from Relay 2 (i.e., the $-$ signal from Relay 1 is reversed at Relay 2); this occurs with probability $0.75 \times 0.25 = 0.1875$, because the relays are assumed to be independent.

Thus,

$$P(B|\overline{A}) = 0.1875 + 0.1875 = 0.375.$$

Finally,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})} = \frac{0.625(0.6)}{0.625(0.6) + 0.375(0.4)} \approx 0.714.$$