

CHAPTER 3 PROBLEMS

1. The zinc-phosphate coating on the threads of steel tubes used in oil and gas wells is critical to their performance. However, 12 percent of all tubes receive an improper amount of coating (either much too low or much too high). That is, about 12 percent of the tubes are defective. Assume that the tubes are independent.

- (a) If we take a sample of 10 tubes, what is the probability that at least 2 are defective?
 (b) If we continually observe tubes until we find the first defective tube, what is the probability that we will observe no more than 3 tubes?
 (c) If we continually observe tubes until we find the second defective tube, what is the probability that we will observe more than 4 tubes?

2. I have recently become involved with a massive ongoing epidemiological investigation, evolving from the Inter-University Convention for the Prevention of AIDS and conducted primarily in West African nations, under the purview of colleagues at the Harvard School of Public Health. In this particular study, the cohort of interest involves female commercial sex workers in Dakar, Senegal. As part of a disease monitoring programme, a total of 1948 female HIV-seronegative workers were recruited and were followed to monitor HIV progression over a 14-year time frame. During the monitoring period, 282 of the workers, initially negative, became positive.

- (a) In order to collect additional demographic information, we will contact 50 of the workers. What is the probability that at least 5 of those contacted are now HIV positive? *Hint:* You can make this calculation under the hypergeometric, the binomial, or the Poisson assumption.
 (b) Treating each sex worker as a “trial,” and treating HIV-infection as a “success,” do you think that the Bernoulli trial assumptions hold here? Why or why not?

3. Let Y be a random variable with mean $\mu = E(Y)$ and variance $\sigma^2 = V(Y)$. The **skewness** associated with Y , denoted by ξ , is given by

$$\xi = \frac{E[(Y - \mu)^3]}{\sigma^3}.$$

The skewness ξ quantifies the level of which a probability distribution departs from symmetry. Thus, if $\xi = 0$, then the distribution associated with Y is said to be **symmetric** in appearance. Values of ξ that are larger (smaller) than zero correspond to distributions that are skewed right (left).

(a) Show that the skewness associated with $Y \sim b(n, p)$ is given by

$$\xi = \frac{1 - 2p}{\sqrt{np(1 - p)}}.$$

Hint: First show

$$E[(Y - \mu)^3] = E(Y^3) - 3\mu E(Y^2) + 2\mu^3.$$

For $Y \sim b(n, p)$, we know $E(Y) = \mu = np$ and $V(Y) = \sigma^2 = np(1 - p)$. Use the mgf to find $E(Y^3)$. You can find $E(Y^2)$ using either the mgf or by noting that $E(Y^2) = \sigma^2 + \mu^2$.

(b) For the $b(n, p)$ model, what does ξ converge to as $n \rightarrow \infty$? Give a practical interpretation of this finding.

4. We have talked about 6 different probability models for a discrete random variable Y ; namely, discrete uniform, binomial, geometric, negative binomial, hypergeometric, and Poisson. Pick **one** topic from the following list:

| | | |
|----------|----------------|---------|
| Sports | Business | Science |
| Politics | Social Science | Arts |

With your chosen topic, give an example of where each probability model might be used. For example, if you choose Business, then you could use the binomial model to count the number of insurance claims filed for a sample of 100 customers. For each of the six models, comment on the assumptions that are needed for your model to be viable, and discuss whether or not these assumptions are plausible.

5. Screening for infectious diseases in a blood-bank setting is a major part of ensuring blood safety. At a local clinic, subjects' blood donations are tested for infection. Suppose that 10 percent of all blood donations are infected in some way (e.g., HIV, gonorrhea, chlamydia, etc.). For simplicity, let's assume that all subjects are independent.

(a) Let W denote the number of subjects tested to find the second infected blood donation. Find $P(W > 3)$.

(b) Suppose that, during a given day, there are 30 donations. Find the probability that no more than two of these donations are infected.

6. Past studies have shown that 1 out of every 10 cars on the road has a speedometer that is miscalibrated. For this problem, assume that different cars are independent and that each has the same $1/10$ probability of being miscalibrated.

(a) In a sample of 10 cars, let X denote the number of cars which are miscalibrated. Write the pmf for X and compute $P(X \geq 3)$. Interpret this probability in words.

(b) Suppose that we continually observe cars until we find the first car with a miscalibrated speedometer. Let Y denote the number of cars that we will observe. Write the pmf for Y and compute $P(Y \leq 6)$. Interpret this probability in words.

(c) Suppose that we continually observe cars until we find the second car with a miscalibrated speedometer. Let Z denote the number of cars that we will observe. Write the pmf for Z and compute $P(Z = 4)$. Interpret this probability in words.

7. A large number of insects are expected to be attracted to a certain variety of rose plant. A commercial insecticide (e.g., Roundup) is advertised as being 99.9 percent effective at killing insects. Suppose that 1,000 insects infest a rose garden where the insecticide has been applied, and let X denote the number of surviving insects (out of 1,000). Define the event $B = \{X \leq 3\}$.

(a) Compute $P(B)$ under a binomial model assumption for X .

(b) Compute $P(B)$ under a Poisson model assumption for X .

(c) Are your answers for (a) and (b) close? Why?

8. Sowbugs are land crustaceans. They are primarily nocturnal, thrive in a moist environment, and they eat decaying leaf litter and vegetable matter. Suppose that Y , the number of sowbugs on a square-foot plot, follows a Poisson distribution with $\lambda = 15$.

- (a) Find the probability that a square-foot plot contains exactly 10 sowbugs.
 (b) In terms of the Poisson pmf, write an expression for $P(Y \geq 100)$. There is no need to evaluate it.
 (c) Suppose that the cost (measured in dollars) incurred from sowbug damage, per square-acre plot, depends on Y in the following way: $C = 0.03Y^2 + 0.05Y$. What is the expected cost of damage?

9. Suppose that Y has a Poisson distribution with mean $\lambda > 0$.

- (a) Derive the mgf of Y . Make sure you explain all the steps.
 (b) For a support point y , use the Poisson pmf to show that

$$\frac{P(Y = y + 1)}{P(Y = y)} = \frac{\lambda}{y + 1}.$$

- (c) What value of y maximizes $P(Y = y)$? This value y is called the **mode** of Y . *Hint:* Consider the result in (b).

10. Suppose that Y has a geometric distribution with success probability p . Let

$$F_Y(y) \equiv P(Y \leq y)$$

denote the cumulative distribution function of Y .

- (a) Show that

$$F_Y(a) = 1 - (1 - p)^a,$$

for any positive integer a .

- (b) Using the result in (a), argue that the geometric distribution possesses the **memoryless property**; i.e., show that, for $b > 0$,

$$P(Y > a + b | Y > a) = P(Y > b).$$

- (c) Show the mgf of Y is given by

$$m_Y(t) = \frac{pe^t}{1 - qe^t},$$

where $q = 1 - p$, for $t < -\ln q$. Make sure you argue why $t < -\ln q$.

11. When circuit boards used in the manufacture of CD players are tested, the percentage of defectives is 5 percent. A sample of 20 circuit boards is available for analysis. Let Y denote the number of defectives (out of 20) and assume a binomial model.

- (a) Briefly, discuss the Bernoulli assumptions in the context of this example. That is, tell me what would have to be true for the binomial model to hold.
 (b) If the sample of 20 circuit boards contains 2 or more defectives, the entire sample is rejected. Find the probability that the sample is rejected.
 (c) In lots of size 20, the cost (in dollars) associated with reworking defective circuit boards is $C = 200Y^2 - 10Y + 100$. Find the expected value of C .

12. Suppose that Y , the number of infected trees distributed on a square-acre plot, follows (approximately) a Poisson distribution with mean $\lambda = 3$.

- (a) What is the probability a single square-acre plot will contain 2 or fewer infected trees?
 (b) Suppose that we continually observe square-acre plots (in a very large forest) until we observe the fourth plot with 2 or fewer infected trees. Let X denote the number of plots we will need to observe. Write down the pdf for X and compute $P(X = 6)$. You may assume that the square-acre plots are independent.
 (c) The time, T , (measured in hours) needed to locate and treat all infected trees is thought to be a linear function of Y . One researcher assumes that $T = 2Y + 3$. Find the mean and variance of T .

13. An animal biologist observes the number of sparrow nests on a 5000 m² plot. Let Y denote the number of sparrow of nests per plot, and assume that Y has a Poisson distribution with mean 2.

- (a) What is the probability that a randomly selected plot contains no sparrow nests?
 (b) Suppose that the biologist observes 10 plots, each of the same size, and let W_1 denote the number of plots that have no sparrow nests on them. Give the distribution of W_1 and find $P(W_1 \leq 1)$. You may assume that the 10 plots are independent.
 (c) Suppose that the biologist continues to observe plots until she finds the first plot that contains no sparrow nests, and let W_2 denote the number of plots she observes. Give the distribution of W_2 and find $P(W_2 = 3)$. You may assume that the plots are independent.
 (d) Suppose that the biologist continues to observe plots until she finds the second plot that contains no sparrow nests, and let W_3 denote the number of plots she observes. What is the (named) distribution of W_3 ? (Just give me the distribution name). You may assume that the plots are independent.

14. A burr is a thin ridge or rough area that occurs when shaping a metal part. During the production process, burrs must be removed using water jets, thermal energy, or some some other method. Assume that Y , the number of burrs present on a part used in automatic transmissions, varies according to a Poisson distribution with mean λ .

- (a) For each part, the cost associated with handling (in \$100s) is given by $C(Y) = a_0 + a_1Y + a_2Y^2$, where a_0 , a_1 , and a_2 are positive constants. Show that the expected cost is given by $E[C(Y)] = a_0 + (a_1 + a_2)\lambda + a_2\lambda^2$.
 (b) Assume that $a_0 = a_1 = 0$, $a_2 = 1$, and $\lambda = 3$ (only for this part of the question). Find the probability that the cost for a certain part is four hundred dollars or less. That is, find $P[C(Y) \leq 4]$.

15. Suppose that $Y \sim \text{binomial}(n, p)$. Show that

$$P(Y > 1 | Y \geq 1) = 1 - \frac{np(1-p)^{n-1}}{1 - (1-p)^n}.$$

16. The number of automobiles entering a mountain tunnel per minute is assumed to follow a Poisson distribution with mean $\lambda = 2$. Starting at 8.00am, you start observing this process. Let T denote the time until the first car enters the tunnel.

- (a) Write out the pdf for T , making sure to note its support.
 (b) What is the probability that you will have to wait 5 minutes or longer to observe the first car entering the tunnel?
 (c) What is the distribution of the time it takes to observe the 13th car entering the tunnel? Simply give me its name.

CHAPTER 4 PROBLEMS

17. In a toxicology experiment, Y denotes the death time (in minutes) for a single rat treated with a certain toxin. The probability density function (pdf) for Y is given by

$$f_Y(y) = \begin{cases} cye^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the value of c that makes this a valid pdf.
(b) Find $P(Y < 5)$ and $P(Y \leq 5)$.

18. Explosive devices used in mining operations produce (nearly) circular craters when detonated. The radii of these craters, say, R , follow an exponential distribution with $\beta = 10$ meters. That is, $R \sim \text{exponential}(10)$. The area of the crater is

$$Y = \pi R^2.$$

- (a) Find the mean area produced by the explosive devices; that is, compute $E(Y)$.
(b) For one detonation, find the probability that the area is greater than 2500 (meters)². That is, compute $P(Y > 2500)$.

19. For a certain class of jet engines, the time (in years) until an overhaul is needed varies according to the following probability density function:

$$f_Y(y) = \begin{cases} ce^{-y/2}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) What is the value of c ?
(b) Compute $P(Y > 2)$, and sketch a pertinent picture.
(c) Derive the cumulative distribution function of Y .
(d) Find the mean and variance of $T = 3Y^2 - 1$.

20. In a chemistry experiment, Y denotes the velocity of a random gas molecule. According to the **Maxwell-Boltzmann** law, the probability density function (pdf) for Y is given by

$$f_Y(y) = \begin{cases} cy^2e^{-y^2/\beta}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where the parameter β is a constant larger than zero.

- (a) Find the value of c that makes this a valid pdf.
(b) Write out an integral expression which equals $P(Y < 1)$. Do not evaluate the integral.

21. Suppose that the random variable X has the pdf

$$f_X(x) = \begin{cases} cx, & 1 < x < 5 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the value of c which makes this a valid pdf, and graph this pdf. Is this a named distribution? If so, which one is it?
 (b) Find the cumulative distribution function $F_X(x)$. Remember that the cdf is defined for all $-\infty < x < \infty$. Graph the cdf.
 (c) Compute $E(X)$ and $V(X)$ without using the mgf.
 (d) Derive the mgf of X .

22. State and prove Chebyshev's Inequality.

23. In the article "Modelling sediment and water column interactions for hydrophobic pollutants" (*Water Research*, **1984**, 1169-1174), the authors model sediment density in a particular region (X , measured in g/cm) as a normal random variable with mean 2.65 and standard deviation 0.85.

- (a) In this region, what is the probability that a sediment specimen exceeds 3.80 g/cm?
 (b) Five percent of all sediment specimens will be below which value?

Draw appropriate pictures for parts (a) and (b).

24. The Weibull distribution is commonly used by engineers to model the time until part failure. The pdf for a Weibull random variable X can be expressed as

$$f_X(x) = \begin{cases} \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

for $\alpha > 0$ and $\beta > 0$. Show that

$$E(X^3) = \Gamma\left(1 + \frac{3}{\alpha}\right) \beta^{3/\alpha}.$$

Hint: Do not try to derive the mgf of X . Compute $E(X^3)$ directly using the pdf.

25. Suppose that Y possesses the probability density function (pdf)

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find $F_Y(y)$, the cumulative distribution function (cdf) of Y , and graph it. Be sure to label your axes.
 (b) Find $E(Y)$.
 (c) Find $P(Y < 1/2)$.

26. Suppose that Y has a gamma distribution with parameters α and β ; i.e., the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

- (a) Use the gamma moment generating function to show that $E(Y) = \alpha\beta$.
 (b) For $m > 1$, show that

$$E(Y^{1/m}) = \frac{\Gamma(\alpha + 1/m)\beta^{1/m}}{\Gamma(\alpha)}.$$

27. Recall that the gamma function is defined, for $\alpha > 0$, as

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy.$$

- (a) If $\alpha > 1$, use an integration by parts argument to show that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
 (b) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. *Hint:* In the integral above, use a substitution with $y = w^2/2$.

28. Suppose Y is a continuous random variable with probability density function (pdf)

$$f_Y(y) = \begin{cases} k(4y - 2y^2), & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the value of k and then graph the pdf.
 (b) Find the cumulative distribution function (cdf) of Y and graph it. Remember to consider the different cases (similarly to how we did it in the notes/HW).
 (c) Compute $P(0.2 < Y < 0.7)$ and $P(Y = 1)$.

Note: I want very detailed graphs.

29. The **median** of a continuous random variable Y with cdf $F_Y(y)$ is the value m such that $F_Y(m) = P(Y \leq m) = 0.5$. That is, Y is just as likely to be larger than its median as it is to be smaller. If $f_Y(y)$ denotes the pdf of Y , then we know that m solves the following equation

$$0.5 = F_Y(m) = \int_{-\infty}^m f_Y(y) dy;$$

that is, the area under $f_Y(y)$ to the left of m is 0.5 and the area to the right of m is also 0.5. For each of the distributions, calculate the median m .

- (a) $Y \sim \mathcal{U}(0, \theta)$
 (b) $Y \sim \text{exponential}(\beta)$
 (c) $Y \sim \mathcal{N}(\mu, \sigma^2)$.

Note: For each of these distributions, graph the pdf and mark the median on the horizontal axis.

30. An environmental engineer at a large gravel company models Y , the monthly gravel sales (in thousands of tons) as a random variable with probability distribution function (pdf)

$$f_Y(y) = \begin{cases} 140y^3(1-y)^3, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Compute $E(Y)$ and $V(Y)$.
 (b) Suppose that the monthly revenue (in \$100,000) from gravel sales is given by $R = 10(1 - Y)$. Find the mean and standard deviation of R .

31. Suppose that Y is a continuous random variable with pdf

$$f_Y(y) = \begin{cases} 2y/\theta^2, & 0 < y < \theta \\ 0, & \text{otherwise.} \end{cases}$$

In this pdf, θ is a positive constant.

- (a) Find $E(Y)$ and $V(Y)$.
 (b) Find the median of Y .

Note: In both parts, your answers should depend on θ .

32. Human papillomavirus (HPV) infection has been established as the cause of virtually all forms of cervical cancer. Large cohort studies incorporate active follow-up with multiple visits and the collection of cervical specimens for HPV DNA testing. This is an important part of disease assessment and allows researchers to collect important covariate information that is linked to disease status. In a recent study, the Guanacaste Project, one of the covariates recorded (for all eligible subjects) was Y , the age at which the subject gave birth to her first child. This variable is possibly linked to HPV infectivity because HPV is a sexually transmitted disease. A normal probability model is attached to Y with mean $\mu = 27.3$ years and standard deviation $\sigma = 3.5$ years. Use this probability model to answer the following questions.

- (a) Find the probability that a mother gives birth to her first child after she is 35 years old.
 (b) Find the proportion of first-time mothers between 20 and 30 years.
 (c) Ten percent of first-time mothers give birth before what age?

Note: In each part, draw detailed picture.

33. Suppose that Y has moment-generating function given by

$$m_Y(t) = \left(\frac{1}{1 - \beta t} \right)^\alpha,$$

for values of $t < 1/\beta$.

- (a) Use the mgf to derive the mean and variance of Y .
 (b) What is the distribution of Y ?

34. During an 8-hour shift, the proportion of time Y that a sheet-metal stamping machine is being serviced for repairs has the following distribution:

$$f_Y(y) = \begin{cases} 2(1 - y), & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the probability that machine repair-time is less than 30 minutes during the 8-hour shift. Sketch a pertinent picture.

- (b) Find the mean and variance of Y .
 (c) The cost C , measured in hundreds of dollars, of the time lost to repairs is given by $C = 10 + 20Y + 4Y^2$. Find $E(C)$ and $V(C)$.

35. Suppose that the random variable Y has the following pdf

$$f_Y(y) = \begin{cases} ke^{-y^2/2}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Note that Y does not have a normal distribution because the support $R = \{y : y > 0\}$. In fact, Y is said to have a **folded-normal** distribution.

- (a) Argue that $k = 2/\sqrt{2\pi}$. *Hint:* For this part, use the fact that the standard normal density is a symmetric function and integrates to 1.
 (b) Show that $E(Y) = 2/\sqrt{2\pi}$
 (c) Show that $V(Y) = 1 - 2/\pi$.

36. Let Y be a random variable with mean $\mu = E(Y)$ and variance $\sigma^2 = V(Y)$. Recall that the **skewness** associated with Y , denoted by ξ , is given by

$$\xi = \frac{E[(Y - \mu)^3]}{\sigma^3}.$$

The skewness ξ quantifies the level of which a probability distribution departs from symmetry. Another measure that describes the distribution of Y is the kurtosis. The **kurtosis** associated with Y , denoted by κ , is given by

$$\kappa = \frac{E[(Y - \mu)^4]}{\sigma^4}.$$

The kurtosis κ measures the “peakedness” of a distribution; i.e., how tall the probability density function is at its peak. A normal random variable Y , for example, has $\kappa = 3$ irrespective of its mean or standard deviation. If a random variable’s kurtosis is greater than 3, it is said to be **leptokurtic**. If its kurtosis is less than 3, it is said to be **platykurtic**. Leptokurtosis is associated with pdfs that are simultaneously “peaked” and have “fat tails.” Platykurtosis is associated with pdfs that are simultaneously less peaked and have thinner tails.

- (a) Derive the skewness and kurtosis associated with $Y \sim \text{exponential}(\beta)$. Is this distribution skewed right or skewed left? Is this distribution leptokurtic or platykurtic? Use your numerical values of ξ and κ to answer this question.

37. In the early stages of drug discovery, it is common to inject rats with a new drug (treatment) and examine them with the hopes of establishing a maximum tolerable dose (MTD) for human subjects. In an early-phase clinical trial (pre-Phase I), a total of 109 rats were injected with a lethal dose of a new drug, and, for each rat, the random variable Y , survival time (in hours), was recorded. The graph below depicts a histogram of the 109 measurements of Y , one for each rat. The physicians ask you to decide on a probability model for the random variable Y . Based on this information, which model would you choose? Be as specific as possible.

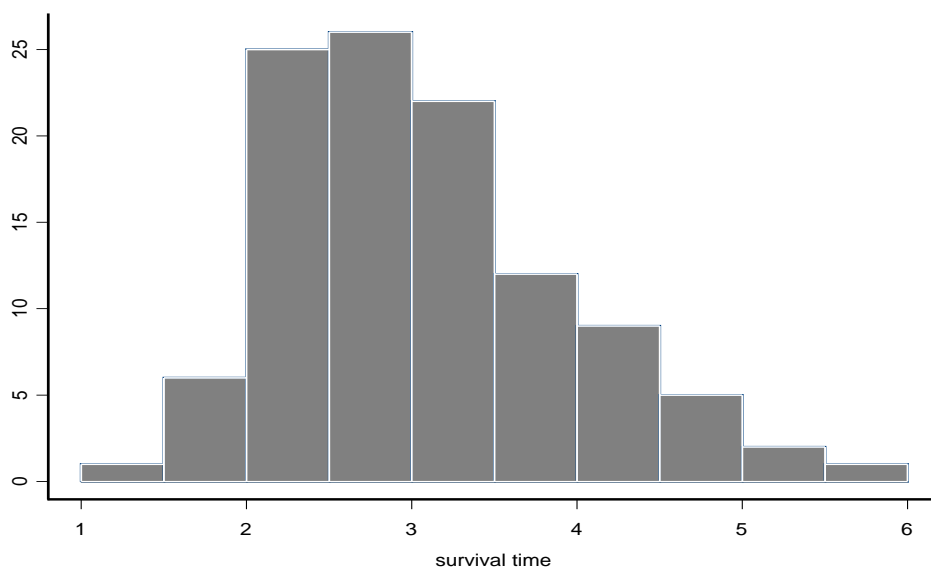


Figure 1: Histogram of $n = 109$ rat survival times.

38. Suppose that Y is a random variable with pdf

$$f_Y(y) = \begin{cases} c(4y - 2y^2), & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

- Show that $c = 3/8$.
- Graph the pdf $f_Y(y)$ over the support of Y .
- Compute $P(Y < 0.75)$; shade in this area under $f_Y(y)$ in your graph from part (b).
- Compute $E(Y)$ and $V(Y)$.
- Compute $E(1/Y)$.

39. Suppose that Y is an exponential random variable with mean $\beta = 1/\lambda$.

(a) Prove that

$$E(Y^k) = \frac{k!}{\lambda^k},$$

for $k = 1, 2, \dots$

(b) The **median** of a random variable Y is the value m such that $F_Y(m) = 0.5$, where $F_Y(\cdot)$ denotes the cdf of Y . Derive a formula for the median of Y , if $Y \sim \text{exponential}(1/\lambda)$.

40. The performance of compressor blades in jet engines is a critical issue to engineers. Historical evidence suggests that for a particular blade, Y , its operational lifetime (in

100s of hours), follows a Weibull distribution with probability density function (pdf)

$$f_Y(y) = \begin{cases} y^2 e^{-y^3/3}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the cumulative distribution function (cdf) for Y and graph it.
 (b) Compute the probability a single blade fails before 100 hours; i.e., compute $P(Y < 100)$.
 (c) Find $E(Y)$ and $V(Y)$. Simplify your answers as much as possible.

Hints: I think you'll find the $u = y^3$ substitution helpful; don't try to derive the mgf.

41. Suppose that $Y \sim \mathcal{U}(\theta_1, \theta_2)$. Derive the moment generating function of Y and use it to derive $E(Y)$ and $V(Y)$. Carefully explain your mgf formula when $t = 0$.

42. For a certain class of new jet engines, the time (in years) until an overhaul is needed varies according to the following probability density function:

$$f_Y(y) = \begin{cases} \frac{1}{4} e^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Derive the cumulative distribution function (cdf) of Y .
 (b) Find the probability that one of these engines will need an overhaul during its first year.
 (c) Find $E(Y)$.
 (d) Find $m_Y(t)$, the moment generating function for Y .

43. To decide on the appropriate premium to charge, insurance companies sometimes use the *exponential principle*, defined as follows. With Y as the random amount that it will have to pay in claims, the premium charged by the insurance company is

$$P = a^{-1} \ln[E(e^{aY})].$$

where a is some specified constant.

(a) Suppose that Y is a gamma(α, β) random variable and let $a = 1/\beta^2$. Under these model assumptions, show that

$$P = \alpha \beta^2 \ln \left(\frac{\beta}{\beta - 1} \right).$$

Graph this resulting relationship as function of β for $\alpha = 1$.

(b) Assuming the same model in part (a), with α fixed, how does P behave for very large values of β ? Investigate this by computing

$$\lim_{\beta \rightarrow \infty} \left[\alpha \beta^2 \ln \left(\frac{\beta}{\beta - 1} \right) \right].$$

Does your answer surprise you? Why or why not?

44. Suppose that Y possesses the probability density function (pdf)

$$f_Y(y) = \begin{cases} y/8, & 0 < y < 4, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find $F_Y(y)$, the cumulative distribution function (cdf) of Y .
(b) Graph the pdf and cdf in two separate graphs, side by side. Be sure to label all axes.

45. Suppose that X is a random variable with pdf

$$f_X(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

- (a) Show that $E(X) = \theta/(\theta + 1)$.
(b) The **median** of a distribution is the value m such that $F_X(m) = \frac{1}{2}$, where F_X denotes the cdf of X . Show that the median of X is

$$m = \left(\frac{1}{2}\right)^{1/\theta}.$$

- (c) It is easy to see that the mean and median will be the same when $\theta = 1$. When this is the case, what is the name of the distribution of X ?