Nonparametric Methods in Reliability

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Abstract

Probabilistic and statistical models for the occurrence of an event of interest over time are described. These models have applicability in the reliability, engineering, biomedical, and other areas where a series of events occur for an experimental unit as time progresses. Nonparametric inference methods, in particular estimation procedures and goodness-of-fit methods, for the model parameters are presented, and their properties described.

Key Words and Phrases: Aalen-Nelson estimator, counting process, frailty, Gaussian process, martingale, minimal repair, perfect repair, product-limit estimator, smooth goodness-of-fit.

1 Introduction

In a variety of situations in reliability, as well as in other sciences, there could be a series of observations made on an experimental unit, with these

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observations representing times to the occurrence of an event of interest. For example, in monitoring a machine (such as a piece of medical equipment), failure of the machine will occur as time progresses, and each time there is a failure, the failed component may be repaired or replaced in order to bring the machine to a functioning state. The process of repairing or replacing a failed component impacts the mechanism for the next failure occurrence.

It is therefore important to have probabilistic and statistical models for the occurrence of the event of interest over time, and there has been active research on this aspect. Furthermore, it is also imperative that methods for making inference about the parameters of these probabilistic models be developed in order to be able to utilize these models for practical purposes, such as predicting the next occurrence of the event, information that could be important regarding safety issues as well as in developing maintenance schedules.

In this paper we discuss several models that have been proposed in modelling the occurrence of an event of interest. These models represent areas where the nonparametric approach is highly useful. These models take into account the type of repair that has been performed after the occurrence of an event. Some methods for making inference, in particular the nonparametric estimation of the model parameters are also presented, and some of the properties of the estimators are described. In Section 2 we present some classes of models that have been put forward, mostly in the reliability area. Section 3 presents inference methods for a general repair model proposed by Dorado, Hollander, and Sethuraman (1997), while Section 4 describes estimation methods for the renewal model when each experimental unit is monitored over a random observation period. A more general model allowing for dependence among the unit's inter-event times, modeled through unobservable frailties, is also considered. Section 5 reviews goodness-of-fit testing of the distribution of the time until the first failure in the Block, Borges, and Savits (1985)'s minimal repair (BBS) model.

2 Repair Models

The construction and analysis of repair models is an important topic because many systems in industrial and health settings are subject to repair after failure. Replacing the failed system by a completely new one is impractical. Thus most repairs restore the system to a status that is not as good as that achieved by replacing the system by a new one. In this section we consider repairable systems (items, units) where, upon failure, the system is repaired in negligible time. We let F denote the distribution of the time to first failure of a new item that is put into operation at time $S_0 = 0$. We let $\bar{F} = 1 - F$ denote the survival function, $\Lambda(t) = \int_0^t dF(s)/\bar{F}(s-)$ denote the (cumulative) hazard function of F and $\lambda(t) = d\Lambda(t)/dt$ denote the failure rate. Let $\{S_j\}_{j \geq 1}$ denote the sequence of failure times and let $T_j = S_j - S_{j-1}$ denote the interfailure times. Repair models can be specified by specifying the joint distribution of the interfailure times. Such joint distributions will depend on F and the nature of the repairs. We list some examples of repair models below.

a. *Perfect Repair or Renewal Model.* Upon failure, the failed system is replaced by a new one stochastically identical to the original so that $T_1, T_2, \ldots$ are independent and identically distributed (iid) according to F.

b. *Minimal Repair Model.* In minimal repair, the system, upon failure, is restored to its state just before failure. Thus if it fails at age t, it is restored to the status of a functioning system of age t. Under this model, $\{S_j\}$ is a Markov process with

$$P\left(S_j > x | S_{j-1} = y\right) = \frac{\bar{F}(x)}{\bar{F}(y)}, \quad x > y.$$

Let $N(t)$ denote the number of failures by time t in a process of minimal repair. Then $N(t)$ is a nonhomogeneous Poisson process with mean value function $E(N(t)) = \Lambda(t)$ and intensity function $\lambda(t)$. For a proof of this result, see Resnick (1994), Section 4.11.

c. *BP Model.* Brown and Proschan (1983) generalized the minimal repair model by allowing two types of repairs. Upon failure, with probability p a perfect repair is performed and with probability $1 - p$ a minimal repair is performed. Denote by F_p the distribution of the time to the first perfect repair and let λ_p denote its failure rate.

Theorem 1 (Brown and Proschan (1983)) *For the BP model, (i)* $\lambda_p(t) = p\lambda(t)$ *and (ii)* $\bar{F}_p(t) = \bar{F}^p(t)$.

Proof: (i) Conditional on no perfect repairs having occurred in $[0, t)$ the item at time t acts as an item of age t, and thus has failure rate $\lambda(t)$. After a failure, a perfect repair is made with probability p. Hence the conditional intensity of a perfect repair at time t, given there have been no perfect repairs in $[0, t)$, is $\lambda_p(t) = p\lambda(t)$.

(ii) $\bar{F}_p(t) = \exp\{-\int_0^t \lambda_p(s)ds\} = \exp\{-\int_0^t p\lambda(s)ds\} = \bar{F}^p(t)$. ■

d. *BBS Model.* Block et al. (1985) generalized the BP model by allowing the probability of a perfect repair to depend on the age of the failed item. In the BBS model $p(\cdot)$ is a measurable function $p : [0, \infty) \to [0, 1]$. BBS showed that for continuous F, the waiting time between perfect repairs is almost-surely finite with distribution H given by

$$H(t) = 1 - \exp\left\{-\int_0^t \frac{p(s)}{\bar{F}(s)} dF(s)\right\}, \quad t \geq 0. \tag{2.1}$$

The generalization of this result to possibly discontinuous F is given by Hollander, Presnell, and Sethuraman (1992).

e. *Kijima Models.* Kijima (1989) introduced models that allow repairs that are better than minimal repairs but not necessarily as good as perfect repairs. Kijima's models restore the repaired item to an effective age that depends on its age just before failure as well as on "degree-of-repair" random variables. We let A_{j+1} denote the effective age of the system after the j^{th} repair with $A_1 \stackrel{\text{def}}{=} 0$. Let D_j, $j = 1, 2, \ldots$, denote the degree of repair random variables. They are assumed to be independently distributed on $[0, 1]$ and independent of other processes.

In Kijima's model I

$$P\left(T_j > x | T_1, \ldots, T_{j-1}, D_1, \cdots, D_{j-1}\right) = \frac{\bar{F}(x + A_j)}{\bar{F}(A_j)}, \tag{2.2}$$

where

$$A_j = \sum_{i=1}^{j-1} D_i T_i, \quad j > 1. \tag{2.3}$$

Thus, in Kijima I, $A_{j+1} = A_j + D_j T_j$.

In Kijima's model II, $P(T_j > x | T_1, \ldots, T_{j-1}, D_1, \ldots, D_{j-1})$ is also given by the right-hand-side of (2.2) but with the specification

$$A_j = \sum_{k=1}^{j-1} \left(\prod_{i=k}^{j-1} D_i \right) T_k, \quad j > 1. \tag{2.4}$$

Thus, in Kijima II, $A_{j+1} = D_j(A_j + T_j)$. With $D_j = 1$ with probability p and $= 0$ with probability $1 - p$, Kijima II reduces to the BP model.

f. *DHS Model*. Dorado et al. (1997) defined a general repair model that contains many of the models in the literature and introduces new models as well. They consider the family of survival functions $\bar{F}_a^\theta(x) = \bar{F}(\theta x + a)/\bar{F}(a)$. The family of distributions $\{F_a^\theta\}$ are stochastically ordered in θ. That is, $\theta \leq \theta'$ implies $F_a^\theta \overset{\text{st}}{\geq} F_a^{\theta'}$, for each a, so that $F_a^\theta(t) \leq F_a^{\theta'}(t)$ for every t. The DHS general repair model depends on two sequences $\{A_j\}_{j\geq 1}$ and $\{\theta_j\}_{j \geq 1}$ known as the effective ages and life supplements, respectively. These sequences satisfy

$$\begin{gathered} A_1 = 0, \quad \theta_1 = 1, \quad A_j \geq 0, \quad \theta_j \in (0, 1], \\ A_j \leq A_{j-1} + \theta_{j-1} T_{j-1}, \quad j \geq 2. \end{gathered} \tag{2.5}$$

The joint distributions of the interfailure times are given as

$$P(T_j \leq t | A_1, \ldots, A_j, \theta_1, \ldots, \theta_j, T_1, \ldots, T_{j-1}) = F_{A_j}^{\theta_j}(t), \tag{2.6}$$

for $t > 0$, $j \geq 1$. From (2.5) and (2.6) we see that for $j \geq 1$, the effective age of the system after the j^{th} repair is less than the effective age $X_j \stackrel{\text{def}}{=} A_j + \theta_j T_j$ just before the j^{th} failure, and since $\theta_j \leq 1$, X_j is less than the actual age S_j.

Some special cases of the DHS model are as follows. If we set $\theta_j = 1$, $A_j = 0$ for $j \geq 1$, we obtain the perfect repair model. If we set $\theta_j = 1$, $A_j = S_{j-1}$, $j \geq 1$, we obtain the minimal repair model. If we set $\theta_j = 1$ for each j and let A_j be defined by (2.3), we obtain the Kijima I model. Setting $\theta_j = 1$ for each j and letting A_j be defined by (2.4), yields the Kijima II model. If we set $\theta_1 = 1$, $A_j = \sum_{i=1}^{j-1} \theta_i T_i$ and $0 < \theta_j < 1$ for $j > 1$ we obtain a model we call the supplemented life repair model. Our use of the term "supplemental life" has the following motivation. If a minimal repair were performed at the time of the first failure, T_2 would have the distribution $F_{T_1}^1$. A longer expected life for T_2 is provided, however, if we use the distribution $F_{T_1}^{\theta_2}$ for some θ_2 satisfying $0 < \theta_2 < 1$. Starting with the distribution $F_{T_1}^{\theta_2}$ for T_2 and applying minimal repair after the second failure, T_3 would have the distribution $F_{A_3}^1$ where $A_3 = T_1 + \theta_2 T_2$. If we seek a longer expected life for T_3, we can use the distribution $F_{A_3}^{\theta_3}$ for some θ_3 satisfying $0 < \theta_3 < 1$. By continuing in this way, we obtain the supplemented life model. Under this model, the system has a larger expected remaining life than it would have under minimal repair.

It is of interest to consider monotonicity properties of the expected interfailure times. Theorem 2, due to Dorado (1995), is a typical result. We first give the definition of a "decreasing mean residual life" distribution. The mean residual life (mrl) function corresponding to F is

$$\varepsilon_F(x) = \left\{ \int_x^\infty \bar{F}(y)dy \right\} / \bar{F}(x).$$

Definition 1 *A failure distribution F is said to be a decreasing mean residual life (DMRL) distribution if the mean, $\varepsilon_F(0)$, is finite and*

$$\varepsilon_F(s) \geq \varepsilon_F(t) \quad \text{for all } 0 \leq s \leq t. \tag{2.7}$$

F is said to be an increasing mean residual life (IMRL) distribution if $\varepsilon_F(0)$ is finite and the inequality in (2.7) is reversed.

Theorem 2 (Dorado (1995)) *Assume, in the DHS model of (2.6), that the $\{\theta_j\}_{j\geq 1}$ and $\{A_j\}_{j\geq 1}$ are increasing sequences and F is DMRL. Then $E(T_j)$ is decreasing in j.*

Proof:

$$\begin{aligned} E(T_j) &= \int_0^\infty P\left(T_j > t\right) dt \\ &= \int_\Omega \int_0^\infty \frac{\bar{F}(\theta_j t + A_j)}{\bar{F}(A_j)} dt dP \\ &\leq \int_\Omega \int_0^\infty \frac{\bar{F}(\theta_j t + A_{j-1})}{\bar{F}(A_{j-1})} dt dP \\ &\leq \int_\Omega \int_0^\infty \frac{\bar{F}(\theta_{j-1} t + A_{j-1})}{\bar{F}(A_{j-1})} dt dP = E\left(T_{j-1}\right). \end{aligned}$$

The first inequality follows from the fact that F is DMRL and that the A_j's are increasing. The second inequality holds since the θ_j's are increasing. ■

g. *Last and Szekli Model.* The restriction $\theta_j \in (0, 1]$ imposed in the DHS model does not allow for deterioration due to repair. Last and Szekli (1998) extended the Kijima II model by allowing the larger range $[0, \infty)$ for the degree-of-repair variables $\{D_i\}$. Values of D_i greater than 1 correspond to deterioration due to repair. Last and Szekli (1998) showed that their repair model contains many proposed in the literature including those proposed by Stadje and Zuckerman (1991) and Baxter, Kijima, and Tortorella (1996).

Doyen and Gaudoin (2002) have pointed out a number of other models that are special cases of the DHS model. For example, Wang and Pham (1996) proposed a model that corresponds to the DHS model with the settings $\theta_i = 1/\alpha^{i-1}$ and $A_i = 0$. The failure process is a quasi-renewal process in that the interfailure times are independent but not identically distributed.

Since the DHS model covers a number of repair models in the literature, as well as many new models not studied in detail, we devote the next section to nonparametric inference for the DHS model.

3 Nonparametric Inference for DHS Model

There is a connection between repair models and censored data models that gives insight for the study of repair models and suggests inference methods that parallel those developed for censored data. Suppose we observe the repair process until a fixed time T. The effective age X_j prior to the j^{th} failure is $A_j + \theta_j(S_j - S_{j-1})$ if $S_j \le T$. If $S_{j-1} \le T < S_j$, we cannot observe X_j and the effective age of the system at time T is $A_j + \theta_j(T - S_{j-1})$, which can be written as $X_j \wedge (A_j + \theta_j(T - S_{j-1}))$, a representation similar to that encountered in censored data. We define the processes

$$N(t) = \sum_j I\left(X_j \le t, S_j \le T\right);$$

$$Y(t) = \sum_j I\left(A_j < t \le \left(X_j \wedge \left[A_j + \theta_j\left(T - S_{j-1}\right)\right]\right)\right).$$

Let $\delta_j = I(S_j \le T)$ and set $\widetilde{X}_j = X_j \wedge [A_j + \theta_j(T - S_{j-1})]$. Then the random variables $\{(\widetilde{X}_1, \delta_1), (\widetilde{X}_2, \delta_2), \ldots\}$ can be thought of as observations coming from a censored model. A repair model observed during $[0, T]$ is similar to a survival study where a subject enters the study at A_j (the system at failure time S_{j-1} is repaired to effective age A_j) and either dies during the study at age X_j (a failure occurs) or leaves the study by $A_j + \theta_j(T - S_{j-1})$ (the system that was repaired at time S_{j-1} has not yet by time T suffered its next failure). From this viewpoint, $N(t)$ is the number of observed (uncensored) deaths by time t and $Y(t)$ is the number at risk at time t.

Next, we define the process

$$M(t) = N(t) - \int_0^t Y(s)\,d\Lambda(s).$$

Typically, analogous to results in censored data theory (Aalen (1978); Fleming and Harrington (1991)), it is natural to try to establish that M is a martingale with respect to the history of N. This proved to be difficult but

Dorado et al. (1997) were able to show that the M process does have the same mean and covariance structure as if it were a martingale. They proved

$$E\left(M(t)\right) = 0; \tag{3.8}$$

$$\operatorname{cov}\left(M(t), M(t')\right) = \int_0^{t \wedge t'} E(Y)(1 - \Delta\Lambda) d\Lambda. \tag{3.9}$$

These results and techniques of Gill (1980) are sufficient to obtain asymptotic properties of the estimator of F. We sketch the development here and refer the reader to Dorado et al. (1997) for details.

We suppose we observe n independent copies of the processes N and Y on a finite interval $[0, T]$, and let N_n and Y_n denote the sum of the first n copies. We wish to estimate F based on these observations. A natural estimator of the failure rate is N_n/Y_n, the ratio of observed deaths at time t to the number at risk at time t. Thus a natural estimator of the cumulative hazard function is the Nelson-Aalen estimator

$$\widehat{\Lambda}_n(t) = \int_0^t \frac{J_n dN_n}{Y_n}$$

where $J_n(t) = I(Y_n(t) > 0)$ for $t \in (0, T]$. It is easy to see that F satisfies

$$F(t) = \int_0^t \left(1 - F(s-)\right) d\Lambda(s)$$

and hence we want an estimator $\widehat{F}_n$ of F to satisfy

$$\widehat{F}_n(t) = \int_0^t \left(1 - \widehat{F}_n(s-)\right) d\widehat{\Lambda}_n(s).$$

The solution of this Volterra integral equation is

$$\widehat{\overline{F}}(t) = \prod_{s \leq t} \left(1 - d\widehat{\Lambda}_n(s)\right)$$

where $\prod_{s\leq t}(1-d\widehat{\Lambda}_n(s))$ denotes the product integral (see Gill and Johansen (1990); Andersen, Borgan, Gill, and Keiding (1993)).

Let $M_n = N_n - \int Y_n d\Lambda$. This is the sum of n iid processes in $D[0,T]$ with mean 0 and covariance function given by (3.9). Thus $W_n(t) = n^{-1/2}M_n(t)$, $0 \leq t \leq T$, will converge to a Gaussian process if tightness can be established. This is done in Theorem 5.1 of Dorado et al. (1997). Dorado et al. (1997) showed that

$$\frac{\widehat{F}_n(t) - F(t)}{\bar{F}(t)} = \int \frac{\bar{\widehat{F}}_n(s-)J_n(s)}{\bar{F}(s)(Y_n(s)/n)} dM_n(s). \tag{3.10}$$

Let

$$C(t) = \int_0^t \frac{dF}{EY(1-F)}.$$

Assume that $F(T) < 1$ and F is an increasing failure rate distribution. From the continuous mapping theorem (see Billingsley (1968)), and a result on the uniform convergence of the integrand in (3.10), Corollary 5.1 of Dorado et al. (1997) shows

$$\sqrt{n}\left(\frac{\widehat{F}_n - F}{\bar{F}}\right) \Rightarrow B(C) \text{ on } D[0,T]$$

where B denotes the Brownian motion on $[0,\infty)$. They also proved

$$\sqrt{n}\frac{\bar{K}}{\bar{F}}(\widehat{F}_n - F) \Rightarrow B^0(K) \text{ on } D[0,T]$$

where B^0 denotes a Brownian bridge on $[0,1]$ and $K = C/(1+C)$.

Dorado et al. (1997) also derived a simultaneous confidence band for F. For $t \in [0,T]$, let $L_n = I(\widehat{F}_n(t) < 1)$ and set

$$\widehat{C}_n(t) = \int_0^t J_n L_n d\widehat{F}_n/[(Y_n/n)(1-\widehat{F}_n)] \text{ and } \widehat{K}_n(t) = \widehat{C}_n(t)/(1+\widehat{C}_n(t)).$$

For t such that $\widehat{F}_n(t) = 1$, set $\widehat{K}_n(t) = 1$. A nonparametric asymptotic simultaneous confidence band for F with confidence coefficient at least $100(1-\alpha)\%$ is

$$\left[\widehat{F}_n \pm n^{-1/2}\lambda_\alpha \widehat{\bar{F}}_n / \widehat{K}_n\right] \tag{3.11}$$

where λ_α is such that $P\left(\sup_{t\in[0,1]}|B^0(t)| \le \lambda_\alpha\right) = 1-\alpha$. Values of λ_α can be obtained from Hall and Wellner (1980) and Koziol and Byar (1975).

Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the distinct ordered values of the X's whose corresponding failure times are within $[0, T]$. Also, let δ_j be the number of observations with value $X_{(j)}$. Then for computational purposes one can use the simplified formulas

$$\widehat{\bar{F}}_n(t) = \prod_{X_{(j)}\le t}\left(1 - \frac{\delta_j}{Y_n(X_{(j)})}\right);$$

$$\widehat{C}_n(t) = n \sum_{X_{(j)}\le t} \frac{\widehat{F}_n(X_{(j)}) - \widehat{F}_n(X_{(j-1)})}{Y_n(X_{(j)})\widehat{\bar{F}}_n(X_{(j)})}.$$

In practice, it may be that the data obtained lead to $\widehat{F}_n(t_0) = 1$ for some $0 < t_0 < T$. When this happens, the data yield a confidence band only on the interval $[0, \sigma)$ where $\sigma = \inf\{t \in [0, T] : \widehat{F}_n(t) = 1\}$.

Gill (1981) considered the testing with replacement scenario where one observes $X_1, X_2, \ldots$ nonnegative iid random variables with distribution F. He derives a nonparametric product limit estimator of F based on the first n of an infinite sequence of independent realizations of

$$\widetilde{N}(t) = \#\left\{j :\ge 1 : \sum_{i=1}^{j} X_i \le t\right\},$$

each observed over a fixed interval $[0, T]$. If we set $A_j = 0$, $\theta_j = 1$ for all j in the DHS model, then the DHS estimator reduces to Gill's estimator and the nonparametric simultaneous confidence band given by DHS provides a band for F in Gill's testing with replacement situation.

4 Renewal Model with Random Termination

Peña, Strawderman, and Hollander (2001) generalized Gill's estimator by allowing each process to be observed over a random time where the times are iid according to a distribution G. Their model postulates that for unit i out of n units, the recurrent event process is observed over the random period $[0, \tau_i]$ where $\tau_i,\ i = 1, 2, \ldots, n$, are iid according to the distribution G. The successive inter-event times $T_{ij}, j = 1, 2, \ldots$, are assumed to be iid from the unknown continuous distribution F. The successive calendar times of event occurences for the ith unit are denoted by

$$0 \equiv S_{i0} < S_{i1} < S_{i2} < S_{i3} < \ldots \quad \text{with} \quad S_{ij} = \sum_{k=1}^{j} T_{ij}.$$

The number of events that occurred on or before calendar time s for unit i is denoted by $N_i^{\dagger}(s)$, so that

$$N_i^{\dagger}(s) = \max\{j \in \{0, 1, 2, \ldots\} : S_{ij} \leq \min(s, \tau_i)\} = \sum_{j=1}^{\infty} I\{S_{ij} \leq \min(s, \tau_i)\}.$$

We denote by $K_i = N_i^{\dagger}(\infty)$, the total number of observed events over $[0, \tau_i]$ for the ith unit. We define doubly-indexed processes for the ith unit by

$$N_i(s, t) = \textstyle\sum_{j=1}^{N_i^{\dagger}(s)} I\{T_{ij} \leq t\} \tag{4.12}$$

$$Y_i(s,t) = \sum_{j=1}^{N_i^\dagger(s-)} I\{T_{ij} \geq t\} + I\{\min(s,\tau_i) - S_{iK_i(s-)} \geq t\}. \quad (4.13)$$

The process $N_i(s,t)$ represents the number of events that occurred on or before time s whose inter-event times are at most t, whereas $Y_i(s,t)$ represents the number of events over $[0,s]$ whose inter-event times are at least t plus a count on whether the right-censored last inter-event time is also at least t. The aggregated processes based on n units are denoted by

$$N(s,t) = \sum_{i=1}^{n} N_i(s,t) \quad \text{and} \quad Y(s,t) = \sum_{i=1}^{n} Y_i(s,t).$$

The resulting product-limit type estimator of $\bar{F} = 1 - F$ based on data that have accrued over the calendar time $[0,s]$ for n units is

$$\hat{\bar{F}}_n(s,t) = \prod_{w=0}^{t} \left[1 - \frac{N(s,dw)}{Y(s,w)}\right]. \quad (4.14)$$

Following ideas of Sellke (1988) and Gill (1980), the following asymptotic properties of this product-limit type estimator were established in Peña et al. (2001).

Theorem 3 *Let $\rho(\cdot) = \sum_{k=1}^{\infty} F^{*k}(\cdot)$ be the renewal function and Λ be the hazard function of F, respectively. Define $G_s(t) = G(t)I\{t < s\} + I\{t \geq s\}$, $\bar{G} = 1 - G$, and*

$$y(s,t) = \bar{F}(t)\bar{G}_s(t-)\left\{1 + \frac{1}{\bar{G}_s(t-)}\int_t^\infty \rho(w-t)dG_s(w)\right\}.$$

Then, if $t^* \in (0,\infty)$ is such that $y(s,t^*) > 0$ and $\Lambda(t^*) < \infty$, as $n \to \infty$,

(i) $\sup_{0 \leq t \leq t^*} \left|\hat{\bar{F}}_n(s,t) - \bar{F}(t)\right|$ *converges in probability to zero;*

(ii) the process $\{W_n(s,t) = \sqrt{n}[\hat{\bar{F}}_n(s,t) - \bar{F}(t)] : 0 \leq t \leq t^*\}$ *converges weakly in Skorohod's space* $\mathcal{D}[0,t^*]$ *to a zero-mean Gaussian process* $\{W^\infty(s,t) : 0 \leq t \leq t^*\}$ *whose covariance function is*

$$Cov\{W^\infty(s,t_1), W^\infty(s,t_2)\} = \bar{F}(t_1)\bar{F}(t_2)\int_0^{\min(t_1,t_2)} \frac{\Lambda(dw)}{y(s,w)}.$$

A possible estimator of the variance of $\hat{\bar{F}}_n(s,t)$ is given by

$$\hat{\sigma}_n^2(s,t) = \hat{\bar{F}}_n(s,t)^2 \int_0^t \frac{N(s,dw)}{Y(s,w)[Y(s,w) - N(s,\Delta w)]}.$$

Together with the weak convergence result, this estimate of the variance could be utilized to form a $100(1-\gamma)\%$ asymptotic confidence interval for $\bar{F}(t)$ given by

$$\left[\hat{\bar{F}}_n(s,t) \pm z_{\gamma/2}\hat{\sigma}_n(s,t)\right],$$

where $z_{\gamma/2}$ is the $100(1-\gamma/2)\%$ percentile of the standard normal distribution. It is still open to develop a simultaneous confidence band in this setting, although by virtue of the form of the limiting covariance function in Theorem 3 a Hall-Wellner (1980) type of band is possible.

Notice that these results are analogous to properties of the product-limit estimator for single-event settings, except that the limiting covariance function in the recurrent event setting now involves the renewal function ρ of the distribution F. The entry of this renewal function in the limiting covariance function is a manifestation of the sum-quota accrual scheme which forces the number of events for the ith unit which were observed over $[0,\tau_i]$

to be informative and makes the censoring mechanism of the last event to be informative as well.

Peña et al. (2001) also considered a model wherein the inter-event times for a unit are correlated. This dependence among the inter-event times is induced by an unobserved latent or frailty variable. To describe this correlated recurrent event model, it is postulated that there is an unobserved Z_i, with $Z_1, Z_2, \ldots, Z_n$ iid random variables from a distribution H_Z, which is taken in particular to be a gamma distribution with mean 1 and variance $1/\alpha$, where $\alpha > 0$ is unknown. Note that the gamma distribution for this frailty variable has the same shape and scale parameter, and this is in order to achieve model identifiability. Given $Z_i = z$, it is assumed that the inter-event times $T_{i1}, T_{i2}, \ldots$ are iid with survivor function

$$\bar{F}(t|z) = P\{T_{ij} > t | Z_i = z\} = [\bar{F}_0(t)]^z. \tag{4.15}$$

This is equivalent to postulating that the conditional hazard function of T_{ij}, given $Z_i = z$, is $\Lambda(t|z) = z\Lambda_0(t)$ where $\Lambda_0 = -\log \bar{F}_0$ is the hazard function of F_0. As a consequence, the joint survivor function of $(T_{i1}, T_{i2}, \ldots, T_{ik})$ for fixed k is given by

$$\begin{aligned} & P\{T_{i1} > t_1, T_{i2} > t_2, \ldots, T_{ik} > t_k\} \\ = \ & \int_0^\infty \left[\prod_{j=1}^k \bar{F}_0(t_j)\right]^z \frac{\alpha^\alpha}{\Gamma(\alpha)} z^{\alpha-1} \exp\{-\alpha z\} dz = \left[\frac{\alpha}{\alpha + \sum_{j=1}^k \Lambda_0(t_j)}\right]^\alpha. \end{aligned}$$

From this, by setting $t_j = t$ and $t_l = 0, l \neq j$, we immediately see that the inter-event times are dependent and the common marginal survivor function of the inter-event times is

$$\bar{F}(t) = P\{T_{ij} > t\} = \left[\frac{\alpha}{\alpha + \Lambda_0(t)}\right]^{\alpha}. \tag{4.16}$$

The semiparametric estimation of this marginal survivor function was discussed in Peña et al. (2001). Mimicking ideas of Nielsen, Gill, Andersen, and Sorensen (1992), the computation of the estimator relies on the expectation-maximization (EM) algorithm (see Dempster, Laird, and Rubin (1977)), where the frailty values are considered as missing values. Given values of $(Z_1, Z_2, \ldots, Z_n)$, say $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n)$, the first part of the M-step of the algorithm is to obtain the conditional estimate of Λ_0 given by

$$\hat{\Lambda}_0(s, t | \hat{z}_1, \ldots, \hat{z}_n) = \int_0^t \frac{\sum_{i=1}^n N_i(s, dw)}{\sum_{i=1}^n \hat{z}_i Y_i(s, w)}.$$

The second part of the M-step of the algorithm is to maximize a marginal likelihood function for α, given values of $\hat{\Lambda}_0$ and $\hat{z}_i$s. To describe this marginal likelihood, define

$$Y_i^{\dagger}(s) = I\{\tau_i \geq s\} \quad \text{and} \quad R_i(s) = s - S_{iN_i^{\dagger}(s-)}.$$

Note that $R_i(s)$ is the backward recurrence time at s. Then, the marginal likelihood for obtaining the estimate of α is given by

$$\begin{aligned} L_F(s; \alpha) &= \prod_{i=1}^n \left\{ \frac{\Gamma(\alpha + N_i^{\dagger}(s))}{\Gamma(\alpha)} \left[\frac{\alpha}{\alpha + \int_0^s Y_i^{\dagger}(v) d\Lambda_0[R_i(v)]}\right]^{\alpha + N_i^{\dagger}(s)} \times \right. \\ &\qquad \left. \left(\prod_{v \leq s} \left[\frac{Y_i^{\dagger}(v) d\Lambda_0[R_i(v)]}{\alpha}\right]^{N_i^{\dagger}(\Delta v)} \right) \right\}. \end{aligned}$$

Given $\hat{\Lambda}_0(s,t|\hat{z}_1,\ldots,\hat{z}_n)$, $d\Lambda_0[R_i(v)]$ is replaced by the jump of $\hat{\Lambda}_0(s,\cdot)$ at $t = R_i(v)$. The maximization of this marginal likelihood with respect to α is facilitated by iterative procedures, such as the Newton-Raphson algorithm. On the other hand, the E-step of the algorithm proceeds by obtaining the values of the Z_is, given $\hat{\alpha}$ and $\hat{\Lambda}_0$ according to the formula

$$\hat{z}_i = \frac{\hat{\alpha} + N_i^\dagger(s)}{\hat{\alpha} + \int_0^s Y_i^\dagger(v) d\hat{\Lambda}_0[s, R_i(v)]}, \quad i = 1, 2, \ldots, n.$$

The E- and M-steps are then iterated alternately until convergence is achieved. Finally, having obtained the estimate $\hat{\Lambda}_0(s,\cdot)$ and $\hat{\alpha}$, the estimate of the marginal survivor function $\bar{F}$ is

$$\hat{\bar{F}}(s,t) = \left[\frac{\hat{\alpha}}{\hat{\alpha} + \hat{\Lambda}_0(s,t)}\right]^{\hat{\alpha}}. \tag{4.17}$$

A competing estimator was that proposed by Wang and Chang (1999), which applies even if the frailty components are not gamma distributed, hence their estimator is more general. In Peña et al. (2001), these two estimators, as well as the estimator which ignored the frailty components, were compared in terms of their bias and mean-squared error functions. It was found that if the gamma frailty model holds, then the semiparametric estimator in (4.17) outperforms the Wang-Chang estimator. The comparisons, which were done through computer simulation studies, also demonstrated that the estimator which ignored the frailty components have a non-negligible systematic bias, hence is not a viable estimator of the marginal survivor function of the inter-event times.

The two estimators of the marginal survivor function discussed above, together with the estimator of Wang and Chang (1999), were illustrated in Peña et al. (2001) using a data set pertaining to small bowel motility found in Aalen and Husebye (1991). This data set which consisted of 19 subjects is depicted in the first plot in Figure 1, which provides the successive MMC periods for each unit. Note that each unit has a right-censored last observation. The object of the study leading to the data set was to estimate the mean length of the migratory motor complex (MMC) period. The plots of the three survivor function estimates are provided in the second plot of Figure 1. As pointed out in Peña et al. (2001), the fact that the three curves are quite close to each other, indicating that there is no need for the frailty component, or equivalently that the renewal assumption is viable. The resulting estimate of α obtained from the EM algorithm was $\hat{\alpha} = 10.18$, which was judged to indicate a weak association among the inter-event times.

5 Goodness-of-Fit for the BBS Model

The inferential procedures in the preceding sections dealt with the estimation of the marginal survivor function of the inter-event times. Another type of problem is to test hypothesis concerning this marginal survivor function. In Agustin and Peña (2001) the problem of testing that the distribution of the time to the occurrence of the first event for a unit under the BBS model (Block et al. (1985)) equals some pre-specified distribution function was considered. The basic idea utilized in constructing the goodness-of-fit procedures relies on an embedding approach with origins from Neyman (1937)'s paper on smooth goodness-of-fit tests (see also Rayner and Best (1989)), and implemented in the context of hazard and failure-time analysis in Peña (1998b,a).

Let $\mathcal{T} = [0, \tau]$, where $\tau \leq \infty$ is known, and consider observing n independent BBS processes each up to their first perfect repair, which occurs at the index ν_j. With $W_{jk}, k = 1, 2, \ldots$, representing the successive (calendar) times in which failures occur for the jth unit, the observables are therefore $\{W_{jk} : 1 \leq j \leq n; 1 \leq k \leq \nu_j\}$. We adopt the stochastic process formulation of the BBS model in Hollander et al. (1992). To proceed, define the multivariate counting process $\mathbf{N}^* = \{(N_1^*(t), \ldots, N_n^*(t)) : t \in \mathcal{T}\}$ with $N_j^*(t) = \sum_{k=1}^{\infty} I\{W_{jk} \leq t\},\ j = 1, \ldots, n$, and the filtration $\mathbf{F}^* = \{\mathcal{F}_t^* : t \in \mathcal{T}\}$ by $\mathcal{F}_t^* = \mathcal{F}_0 \vee \bigvee_{j=1}^n \mathcal{F}_{jt}^*$, where $\mathcal{F}_{jt}^* = \sigma\left\{\{N_j^*(s) : s \leq t\} \cup \{U_{jk} : k \geq 1\}\right\}$, and with $\mathcal{F}_0$ containing all null sets of $\mathcal{F}$. The U_{jk}'s are iid standard uniform variates, with U_{jk} determining whether a minimal or a perfect repair is performed after failure at W_{jk}. The relevant multivariate counting process is $\mathbf{N} = \{(N_1(t), \ldots, N_n(t)) : t \in \mathcal{T}\}$ with

$$N_j(t) = N_j^*(t \wedge W_{j\nu_j}), \quad j = 1, \ldots, n,$$

and the corresponding observable filtration $\mathbf{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ is given by $\mathcal{F}_t = \bigvee_{j=1}^n \mathcal{F}_{j(t \wedge W_{j\nu_j})}^*$. The $\mathbf{F}$-compensator of $\mathbf{N}$ is $\mathbf{A} = \{(A_1(t), \ldots, A_n(t)) : t \in \mathcal{T}\}$ with

$$A_j(t) = \int_0^t Y_j(s)\lambda(s)\,\mathrm{d}s, \quad j = 1, \ldots, n,$$

where $Y_j(s) = I\{W_{j\nu_j} \geq s\}$ and $\lambda(\cdot)$ is the unknown baseline hazard function associated with F, the distribution of the time to the first event.. Denote by $\lambda_0(\cdot)$ the hazard rate associated with $F_0(\cdot)$, the hypothesized distribution function. The goodness-of-fit problem is to test the null hypothesis

$$H_0 : \lambda(\cdot) = \lambda_0(\cdot) \quad \text{versus} \quad H_1 : \lambda(\cdot) \neq \lambda_0(\cdot).$$

The main idea in developing the test is to embed the hypothesized hazard rate $\lambda_0(\cdot)$ into a larger parametric family of hazard rate functions. This family is obtained by smoothly transforming $\lambda_0(\cdot)$. Let us define the family of order k smooth alternatives via

$$\mathcal{A}_k = \{\lambda_k(\cdot;\boldsymbol{\theta}) = \lambda_0(\cdot)\exp[\boldsymbol{\theta}'\boldsymbol{\Psi}(\cdot)] : \boldsymbol{\theta} \in \mathbb{R}^k\}, \tag{5.18}$$

where k is some fixed positive integer, and $\boldsymbol{\Psi}(\cdot) = (\psi_1(\cdot), \psi_2(\cdot), \ldots, \psi_k(\cdot))'$ is a $k \times 1$ vector of locally bounded predictable processes. Note that by setting $\boldsymbol{\theta} = \mathbf{0}$ in (5.18), the hypothesized hazard rate function is recovered. Hence, the null hypothesis $H_0 : \lambda(\cdot) = \lambda_0(\cdot)$ can be restated as $H_0^* : \boldsymbol{\theta} = \mathbf{0}$. To derive the score test for this hypothesis, we obtain the score process associated with $\boldsymbol{\theta}$. Under the model in (5.18), the compensator of $\mathbf{N}(\cdot)$ is $\mathbf{A}(\cdot;\boldsymbol{\theta}) = (A_1(\cdot;\boldsymbol{\theta}), \ldots, A_n(\cdot;\boldsymbol{\theta}))$, where

$$A_j(\cdot;\boldsymbol{\theta}) = \int_0^{\cdot} Y_j(s)\lambda_0(s)\exp[\boldsymbol{\theta}'\boldsymbol{\Psi}(s)]\,\mathrm{d}s.$$

From the relevant partial likelihood process obtained via Jacod (1975)'s formula (see Andersen et al. (1993)), straightforward derivation leads to the score process

$$\boldsymbol{U}_{\boldsymbol{\theta}}(t;\boldsymbol{\theta}) = \sum_{j=1}^{n} \int_0^t \boldsymbol{\Psi}(s)\ \mathrm{d}M_j(s;\boldsymbol{\theta}),$$

where $M_j(s;\boldsymbol{\theta}) = N_j(s) - A_j(s;\boldsymbol{\theta}),\ j = 1, 2, \ldots, n.$

To obtain the score test procedure, the distribution of $\boldsymbol{U}_{\boldsymbol{\theta}}(t;\boldsymbol{\theta})$ under the null hypothesis is required. In Agustin and Peña (2001) it was shown, using Rebolledo's martingale central limit theorem (see Andersen et al. (1993)), that under the conditions

(C1) $\int_0^\tau \lambda_0(s)\ \mathrm{d}s < \infty$;

(C2) There exists a $k \times k$ matrix function $\boldsymbol{D}$ such that as $n \to \infty$,

$$\sup_{t\in\mathcal{T}} \left\| \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t)' Y_j(t) - \boldsymbol{D}(t) \right\| \xrightarrow{\text{pr}} 0;$$

(C3) The matrix $\boldsymbol{\Sigma}(\tau) = \int_0^\tau \boldsymbol{D}(t)\lambda_0(t)\ \mathrm{d}t$ is positive definite;

(C4) For each $\epsilon > 0$, $\ell = 1, \ldots, k$, and for every $t \in \mathcal{T}$,

$$\frac{1}{n} \sum_{j=1}^{n} \int_0^t \psi_\ell(s)^2 I\{|\psi_\ell(s)| \geq \sqrt{n}\epsilon\} Y_j(s)\lambda_0(s)\ \mathrm{d}s \xrightarrow{\text{pr}} 0,$$

we have the following asymptotic result.

Theorem 4 *Under the BBS model and* $H_0^*: \boldsymbol{\theta} = \boldsymbol{0}$*, as* $n \to \infty$*,*

$$n^{-1/2}\boldsymbol{U}_{\boldsymbol{\theta}}(\tau;\boldsymbol{0}) \xrightarrow{\text{d}} N(\boldsymbol{0}, \boldsymbol{\Sigma}(\tau)).$$

We focus on the case where $\boldsymbol{\Psi}(\cdot)$ is deterministic. By Glivenko-Cantelli Theorem, it is seen that the limiting covariance matrix is

$$\boldsymbol{\Sigma}(\tau) = \int_0^\tau \boldsymbol{\Psi}(s)\boldsymbol{\Psi}(s)' \exp\left[-\int_0^s p(u)\lambda_0(u)\ \mathrm{d}u\right] \lambda_0(s)\ \mathrm{d}s.$$

Since the probability of perfect repair $p(\cdot)$ is unknown, we need to estimate $\mathbf{\Sigma}(\tau)$. A possible consistent estimator of this is

$$\hat{\mathbf{\Sigma}}(\tau) = \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau \mathbf{\Psi}(s)\mathbf{\Psi}(s)' Y_j(s)\lambda_0(s)\ \mathrm{d}s.$$

By virtue of these results, an asymptotic α-level (smooth) goodness-of-fit test of H_0 versus H_1 is:

$$\textit{Reject } H_0 \textit{ if } S(\tau) \equiv \tfrac{1}{n}\boldsymbol{U}_\theta(\tau;\mathbf{0})'\mathbf{\Sigma}^-(\tau)\boldsymbol{U}_\theta(\tau;\mathbf{0}) \ \geq \ \chi^2_{k^*;\alpha},$$

where $\mathbf{\Sigma}^-(\cdot)$ is a generalized inverse of $\mathbf{\Sigma}(\cdot)$ and $\chi^2_{k^*;\alpha}$ is the $(1-\alpha)100^{\text{th}}$ percentile of the chi-square distribution with degrees of freedom $k^* = \text{rank}[\mathbf{\Sigma}(\tau)]$.

It is apparent in the form of the test statistic that the choice of the process $\mathbf{\Psi}(\cdot)$ is crucial. In fact, $\mathbf{\Psi}(\cdot)$ determines the family of alternatives for which the test will have good power. We focus on a polynomial specification of this process. We consider the specification of $\mathbf{\Psi}(\cdot)$ given by

$$\mathbf{\Psi}(t : \text{PW}_k) = [1,\ \Lambda_0(t),\ \ldots,\ \Lambda_0(t)^{k-1}]', \tag{5.19}$$

where $k \in \{1, 2, \ldots\}$ is a specified order and $\Lambda_0(t) = \int_0^t \lambda_0(s)\mathrm{d}s$. The label PW_k is adopted to distinguish this polynomial specification from other forms of $\mathbf{\Psi}$ explored in Peña (1998b,a) and Agustin and Peña (1999).

The process $\mathbf{\Psi}(\cdot)$ in (5.19) yields the score statistic vector

$$\frac{1}{\sqrt{n}}\boldsymbol{U}_{\boldsymbol{\theta}}(\tau;\mathbf{0}) \equiv \boldsymbol{Q}(\tau : \text{PW}_k) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left[\sum_{i=1}^{N_j(\tau)}(R_{ji})^{\ell-1} - \frac{(R^\tau_{j\nu_j})^\ell}{\ell}\right]_{\ell=1,\ldots,k},$$

where $R_{ji} = \Lambda_0(W_{ji}), (i = 1, 2, \ldots, \nu_j)$ and $R^\tau_{j\nu_j} = \Lambda_0(\tau \wedge W_{j\nu_j})$. The R_{ij}s and $R^\tau_{i\nu_j}$s are generalized residuals (cf., Cox and Snell (1968)) in this BBS model. The limiting covariance matrix is obtained via

$$\boldsymbol{\Sigma}(\tau : \mathrm{PW}_k) = \left[\left(\int_0^\tau \Lambda_0(t)^{\ell+\ell'-2} \exp\{-\Lambda_0^*(t)\}\, \mathrm{d}\Lambda_0(t)\right)_{\ell,\ell'=1,\ldots,k}\right],$$

where $\Lambda_0^*(t) = \int_0^t p(u)\lambda_0(u)\mathrm{d}u$. A consistent estimator of the limiting covariance matrix is

$$\hat{\boldsymbol{\Sigma}}(\tau : \mathrm{PW}_k) = \frac{1}{n}\sum_{j=1}^n \left[\left(\frac{(R_{j\nu_j}^\tau)^{\ell+\ell'-1}}{\ell+\ell'-1}\right)_{\ell,\ell'=1,\ldots,k}\right].$$

Consequently, the asymptotic α-level "polynomial" test of H_0 becomes:

Reject H_0 if $S(\tau : \mathrm{PW}_k) \equiv \boldsymbol{Q}(\tau : \mathrm{PW}_k)'\hat{\boldsymbol{\Sigma}}(\tau : \mathrm{PW}_k)^-\boldsymbol{Q}(\tau : \mathrm{PW}_k) \geq \chi^2_{k;\alpha}$.

We demonstrate a few special cases of this test. If the smoothing order is $k = 1$, we obtain the test statistic

$$S(\tau : \mathrm{PW}_1) = \frac{\left\{\sum_{j=1}^n [N_j(\tau) - R_{j\nu_j}^\tau]\right\}^2}{\sum_{j=1}^n R_{j\nu_j}^\tau}. \qquad (5.20)$$

This is a generalization of the Pearson-type test statistic studied by Akritas (1988). Furthermore, suppose we allow for right-censoring and set $p(t) = 1$, which results in the randomly right-censored model without covariates. Denote the minimum of the failure time and the censoring variable for the j^{th} unit by Z_j, and let δ_j be the correponding censoring indicator. If there are no ties among the Z_j's, then (5.20) simplifies to

$$S(\tau : \mathrm{PW}_1) = \frac{\left[\sum_{j=1}^n (\delta_j - R_j^\tau)\right]^2}{\sum_{j=1}^n R_j^\tau}, \qquad (5.21)$$

where $R_j^\tau = \Lambda_0(Z_j \wedge \tau)$. The statistic in (5.21) is that of Hyde (1977) for right-censored data.

An added bonus to this development of goodness-of-fit tests is that the individual components of $S(\tau : \mathrm{PW}_k)$ are asymptotically χ_1^2-distributed and can be used as directional tests. For $i = 1, \ldots, k$, the i^{th} directional test statistic is

$$S_i(\tau : \mathrm{PW}_k) = \frac{Q_i^2(\tau : \mathrm{PW}_k)}{\hat{\sigma}_i^2(\tau : \mathrm{PW}_k)},$$

where $\hat{\sigma}_i^2(\tau : \mathrm{PW}_k)$ is the $(i,i)^{\mathrm{th}}$ element of $\hat{\mathbf{\Sigma}}(\tau : \mathrm{PW}_k)$. Note that these directional test statistics need not be independent of each other.

If one desires asymptotically independent directional tests, an alternative choice for the $\mathbf{\Psi}(\cdot)$ process is obtained by replacing the polynomial-type specification by orthogonal polynomials. In the classical density-based formulation, Neyman (1937) obtained orthogonal polynomials by choosing the components of $\mathbf{\Psi}$ to be orthonormal with respect to the density specified under the null hypothesis. In the hazard-based formulation, this corresponds to choosing the vector $\mathbf{\Psi}$ such that

$$\int_0^{\tau} \mathbf{\Psi}(w)\mathbf{\Psi}(w)' \exp\left(-\int_0^{w} p(u)\lambda_0(u)\ \mathrm{d}u\right)\lambda_0(w)\ \mathrm{d}w = \boldsymbol{I}_k,$$

where $\boldsymbol{I}_k$ is the identity matrix of order k. In the case of a constant probability of perfect repair, i.e., $p(t) \equiv p$, then the vector of interest is $\mathbf{\Psi}^*$ which satisfies the condition

$$\int_0^{\Lambda_0(\tau)} \mathbf{\Psi}^*(w)\mathbf{\Psi}^*(w)' \exp(-pw)\ \mathrm{d}w = \boldsymbol{I}_k.$$

The Gram-Schmidt orthogonalization procedure may be applied to obtain the elements of $\mathbf{\Psi}^*$. In the limiting case $\tau \to \infty$, the Gram-Schmidt procedure produces

$$\psi_i^*(w) = (-1)^{i-1}\sqrt{p}\sum_{\ell=0}^{i-1}\left(\begin{array}{c} i-1 \\ \ell \end{array}\right)\frac{(-wp)^\ell}{\ell!}, \quad i = 1, \ldots, k, \tag{5.22}$$

where $\left(\begin{array}{c} i \\ \ell \end{array}\right)$ is the combination of i objects taken ℓ at a time. The functions in (5.22) are the scaled Laguerre polynomials. Note that $\psi_i^*(\cdot)$ depends on the unknown parameter p, which needs to be estimated. A consistent estimator of p is $\hat{p} = n/N_\bullet$, where $N_j = N_j(\infty)$ and $N_\bullet = \sum_{j=1}^{n} N_j$. Consequently, the score statistic is $\boldsymbol{Q}(\tau; \mathrm{OR}_k) = (Q_1(\tau; \mathrm{OR}_k), Q_2(\tau; \mathrm{OR}_k), \ldots, Q_k(\tau; \mathrm{OR}_k))'$ with

$$\begin{aligned} Q_h(\tau; \mathrm{OR}_k) &= \frac{1}{\sqrt{n}}(-1)^{h-1}\sqrt{\hat{p}}\times \\ &\sum_{\ell=0}^{h-1}\left\{\left(\begin{array}{c} h-1 \\ \ell \end{array}\right)\frac{(-\hat{p})^\ell}{\ell!}\sum_{j=1}^{n}\left[\sum_{i=1}^{N_j}(R_{ji})^\ell - \frac{(R_{j\nu_j}^\tau)^{\ell+1}}{\ell+1}\right]\right\}. \end{aligned}$$

What do we gain by using orthogonal polynomials? As in the case of the "polynomial" test, the components of $S(\mathrm{OR}_k)$ are χ_1^2-distributed and can be used as directional tests. Whereas the directional tests based on the polynomial specification are asymptotically dependent, the component test statistics from this orthogonal specification are asymptotically independent. Furthermore, simulation studies in Agustin and Peña (2001) demonstrated that the directional components of the test based on the polynomial specification tend to be anticonservative when the sample size is small, in contrast to the behavior of the directional components of the test based on the orthogonal specification. Also, each component of the test based on the orthogonal specification has the potential of detecting specific departures from the hypothesized hazard rate. To cite an example, the simulation results in Agustin and Peña (2001) showed that $S_1(\tau; \mathrm{OR}_4)$ is sensitive against scale changes, while the other components are not sensitive to these alternatives.

There are many other issues not discussed in this review pertaining to this goodness-of-fit problem. For instance, there is the issue of how to choose the smoothing order k in a data-dependent or adaptive manner. The resolution of this problem is still incomplete, but work is in progress. Furthermore, there is the problem of testing that the distribution to the first failure in this BBS model belongs to some pre-specified parametric family of distributions, which is the situation of a composite null hypothesis. This case has been dealt with in Agustin and Peña (2003) in the context of a generalized BBS model.

References

Aalen, O. (1978), "Nonparametric inference for a family of counting processes," *Annals of Statistics*, 6, 701–726.

Aalen, O. and Husebye, E. (1991), "Statistical analysis of repeated events forming renewal processes," *Statistics in Medicine*, 10, 1227–1240.

Agustin, M. Z. N. and Peña, E. A. (1999), "Order statistic properties, random generation, and goodness-of-fit testing for a minimal repair model," *J. Amer. Statist. Assoc.*, 94, 266–272.

— (2001), "Goodness-of-fit of the distribution of time-to-first-occurrence in recurrent event models," *Lifetime Data Anal.*, 7, 289–306.

Agustin, Z. and Peña, E. (2003), "A Basis Approach to Goodness-of-Fit Testing in Recurrent Event Models," *Tentatively accepted by JSPI.*

Akritas, M. (1988), "Pearson-type goodness-of-fit tests: the univariate case," *Journal of the American Statistical Association*, 83, 222–230.

Andersen, P., Borgan, O., Gill, R., and Keiding, N. (1993), *Statistical Models Based on Counting Processes*, New York: Springer-Verlag.

Baxter, L., Kijima, M., and Tortorella, M. (1996), "A point process model for the reliability of a maintained system subject to general repair," *Stochastic Models*, 12, 37–65.

Billingsley, P. (1968), *Convergence of Probability Measures*, New York: John Wiley & Sons.

Block, H., Borges, W., and Savits, T. (1985), "Age-dependent minimal repair," *J. Appl. Prob.*, 22, 51–57.

Brown, M. and Proschan, F. (1983), "Imperfect repair," *J. Appl. Prob.*, 20, 851–859.

Cox, D. and Snell, E. (1968), "A general definition of residuals," *Journal of the Royal Statistical Society*, 30, 248–275.

Dempster, A., Laird, N., and Rubin, D. (1977), "Maximum likelihood estimation from incomplete data via the EM algorithm (with discussion)," *J. Roy. Statist. Soc. B*, 39, 1–38.

Dorado, C. (1995), "On a General Repair Model for Repairable Systems," Ph.D. thesis, Florida State University.

Dorado, C., Hollander, M., and Sethuraman, J. (1997), "Nonparametric estimation for a general repair model," *Ann. Statist.*, 25, 1140–1160.

Doyen, L. and Gaudoin, O. (2002), "Models for assessing maintenance efficiency," In *Mathematical Methods in Reliability: Communications of the 3rd International Conference,* Eds. Langseth, H. and Lindqvist, B., 207–210.

Fleming, T. and Harrington, D. (1991), *Counting Processes and Survival Analysis*, New York: Wiley.

Gill, R. (1980), "Nonparametric Estimation Based on Censored Observations of a Markov Renewal Process," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 53, 97–116.

Gill, R. and Johansen, S. (1990), "A survey of product-integration with a view toward application in survival analysis," *Ann. Statist.*, 18, 1501–1555.

Gill, R. D. (1981), "Testing with replacement and the product-limit estimator," *The Annals of Statistics*, 9, 853–860.

Hall, W. and Wellner, J. (1980), "Confidence bands for a survival curve from censored data," *Biometrika*, 67, 133–143.

Hollander, M., Presnell, B., and Sethuraman, J. (1992), "Nonparametric methods for imperfect repair models," *Ann. Statist.*, 20, 879–896.

Hyde, J. (1977), "Testing survival under right censoring and left truncation," *Biometrika*, 64, 225–230.

Jacod, J. (1975), "Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales," *Z. Wahrsch. verw. Geb*, 31, 235–253.

Kijima, M. (1989), "Some results for repairable systems with general repair," *J. Appl. Prob.*, 26, 89–102.

Koziol, J. and Byar, D. (1975), "Percentage points of the asymptotic distributions of one and two sample K-S statistics for truncated or censored data," *Technometrics*, 17, 507–510.

Last, G. and Szekli, R. (1998), "Asymptotic and monotonicity properties of some repairable systems," *Adv. in Appl. Probab.*, 30, 1089–1110.

Neyman, J. (1937), ""Smooth test" for goodness of fit," *Skand. Aktuarietidskrift*, 20, 149–199.

Nielsen, G., Gill, R., Andersen, P., and Sorensen, T. (1992), "A counting process approach to maximum likelihood estimation in frailty models," *Scand. J. Statist.*, 19, 25–43.

Peña, E. A. (1998a), "Smooth goodness-of-fit tests for composite hypothesis in hazard based models," *Ann. Statist.*, 26, 1935–1971.

— (1998b), "Smooth goodness-of-fit tests for the baseline hazard in Cox's proportional hazards model," *J. Amer. Statist. Assoc.*, 93, 673–692.

Peña, E. A., Strawderman, R. L., and Hollander, M. (2001), "Nonparametric estimation with recurrent event data," *J. Amer. Statist. Assoc.*, 96, 1299–1315.

Rayner, J. and Best, D. (1989), *Smooth Tests of Goodness of Fit*, New York: Oxford University Press.

Resnick, S. (1994), *Adventures in Stochastic Processes*, Boston: Birkhauser.

Sellke, T. (1988), "Weak convergence of the Aalen estimator for a censored renewal process," In *Statistical Decision Theory and Related Topics IV* (eds., S. Gupta and J. Berger), 2, 183–194.

Stadje, W. and Zuckerman, D. (1991), "Optimal maintenance strategies for repairable systems with general degree of repair," *J. Appl. Prob.*, 28, 384–396.

Wang, H. and Pham, H. (1996), "A quasi-renewal process and its application in imperfect maintenance," *International Journal of System Science*, 27, 1055–1062.

Wang, M. C. and Chang, S. H. (1999), "Nonparametric estimation of a recurrent survival function," *Journal of the American Statistical Association*, 94, 146–153.

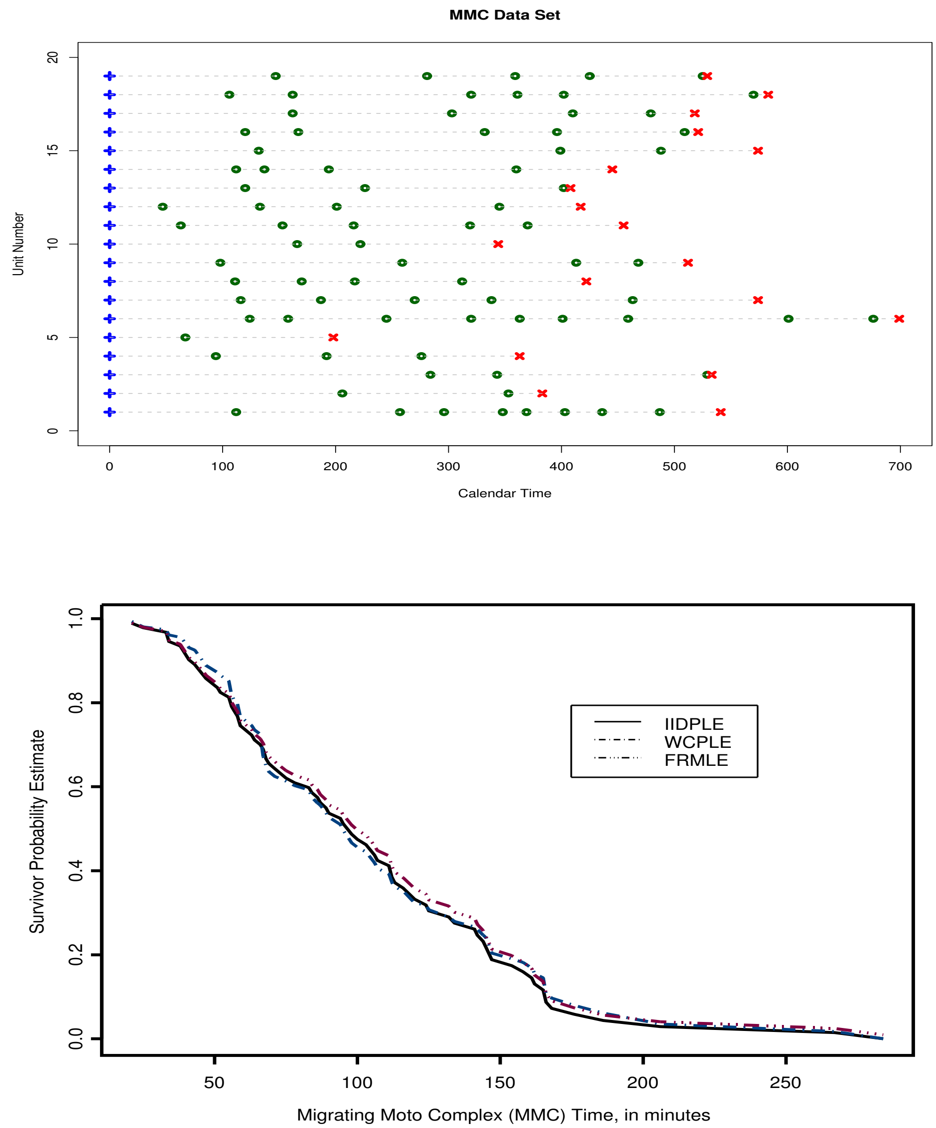

Figure 1: Pictorial representation of the MMC data set and plots of the three survivor function estimates for this data set. IIDPLE is the estimate obtained by assuming the no-frailty renewal model, WCPLE is the estimate of Wang and Chang (1999), and FRMLE is the gamma frailty-based semiparametric estimate. The maximum likelihood estimate of the frailty parameter α under the gamma frailty model is $\hat{\alpha} = 10.17562$, or, equivalently, $\hat{\xi} = \hat{\alpha}/(1+\hat{\alpha}) = 0.9105$.