A Basis Approach to Goodness-of-Fit Testing in Recurrent Event Models

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Abstract

A class of tests for the hypothesis that the baseline hazard function in Cox's proportional hazards model and for a general recurrent event model belongs to a parametric family $\mathcal{C} \equiv \{\lambda_0(\cdot;\xi) : \xi \in \Xi\}$ is proposed. Finite properties of the tests are examined via simulations, while asymptotic properties of the tests under a contiguous sequence of local alternatives are studied theoretically. An application of the tests to the general recurrent event model, which is an extended minimal repair model admitting covariates, is demonstrated. In addition, two real data sets are used to illustrate the applicability of the proposed tests.

Key Words: Counting process; goodness-of-fit test; minimal repair model; Neyman's test; nonhomogeneous Poisson process; repairable system; score test.

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1. Introduction and Setting

Goodness-of-fit testing is an integral part of statistical modeling and has pervaded statistical theory and practice since Pearson's (1900) pioneering paper. In this paper we propose a class of goodness-of-fit procedures for testing whether the baseline hazard function in a Cox's (1972) proportional hazards model (CPHM) and a general recurrent event model belongs to a specified parametric family. In dealing with the CPHM, attention is mostly on the vector of regression coefficients; however, when one is interested in prediction and model validation, then inference on the baseline hazard function becomes necessary. The problem of goodness-of-fit pertaining to the baseline hazard function will be our focus in this paper. In particular, interest is on

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testing whether the baseline hazard function belongs to a parametric family of hazard functions, an important issue for instance in certain cancer studies where there are substantive reasons for testing whether the baseline hazard function belongs to the Weibull class. One advantage in knowing that the baseline hazard belongs to a parametric class is it enables the adoption of inferential methods which are more efficient relative to those which are purely nonparametric. This issue has been discussed for example in Lin and Spiekerman (1996) and Ying, Wei, and Lin (1992), among others.

We seek to develop goodness-of-fit procedures that possess optimality properties. This leads to three possible approaches: likelihood ratio tests, Wald tests, or score tests. Rayner and Best (1989) present an interesting discussion of the properties of these tests. In particular, they conclude that though all three tests are asymptotically equivalent, the score test is usually the simplest to implement. Therefore, we focus on the development of a goodness-of-fit score test which is a hazard-based adaptation of Neyman's (1937) smooth goodness-of-fit tests.

Neyman's (1937) smooth goodness-of-fit tests were derived by embedding the hypothesized class of density functions in a wider class whose members are obtained as "smooth" transformations from the members of the hypothesized class (cf., Rayner and Best, 1989). Recent papers dealing with smooth goodness-of-fit tests are Bickel and Ritov (1992), Eubank and Hart (1992), Inglot, et al. (1994), Ledwina (1994), Kallenberg and Ledwina (1995), and Fan (1996). Gray and Pierce (1985) proposed an extension of this class of tests in the presence of right-censored data, but some difficulties arose when this embedding is through the density functions. With the advent of the modern stochastic process approach ushered by Aalen's (1978) paper in dealing with failure-time models, hazard functions have proven to be the natural parameter in this framework. Within this framework, Neyman's smooth tests were reformulated through hazard functions in Peña (1998a) for the simple hypothesis case in the CPHM with right-censored data, and in Peña (1998b) for the composite hypothesis case in a no-covariate setting. This hazard approach eliminated some technical difficulties encountered using the density embedding, and allowed the use of counting process and martingale theory to resolve technical issues. The present paper further generalizes results in Peña (1998ab) by considering the composite hypothesis testing problem for the CPHM and a general recurrent event model. The mathematical framework of this paper will therefore be counting processes and martingales (cf., Fleming and Harrington (1991); Andersen, Borgan, Gill and Keiding (1993)).

Consider a multivariate counting process $\boldsymbol{N} = \{(N_1(t), N_2(t), \ldots, N_n(t)) : t \in \mathcal{T}\}$ defined on a filtered probability space $(\Omega, \boldsymbol{F} = (\mathcal{F}_t : t \geq 0), \mathcal{P})$. The time index $\mathcal{T}$ maybe $(0, \tau)$ or $(0, \tau]$, $\tau \leq \infty$ and the filtration is typically the natural filtration. For the CPHM or the multiplicative intensity model (see Andersen and Gill (1982)), the $\boldsymbol{F}$-compensator of $\boldsymbol{N}$ is $\boldsymbol{A} = \{(A_1(t; \xi, \beta), A_2(t; \xi, \beta), \ldots, A_n(t; \xi, \beta)) : t \in \mathcal{T}\}$ with

$$A_j(t; \xi, \beta) = \int_0^t Y_j(s) \lambda(s; \xi) \exp\{\beta' \boldsymbol{X}_j(s)\} \, \mathrm{d}s,$$

where $\boldsymbol{Y} = \{(Y_1(t), Y_2(t), \ldots, Y_n(t)) : t \in \mathcal{T}\}$ is a vector of predictable "at-risk" processes, $\lambda(\cdot; \xi)$ is some baseline hazard function, β is a $q \times 1$ vector of regression coefficients, and $\boldsymbol{X}_1(\cdot), \boldsymbol{X}_2(\cdot), \ldots, \boldsymbol{X}_n(\cdot)$ are $q \times 1$ vectors of locally bounded predictable processes. The goodness-of-fit problem is to test the composite null hypothesis $H_0 : \lambda(\cdot) \in \mathcal{C} \equiv \{\lambda_0(\cdot; \xi) : \xi \in \Xi\}$ versus the alternative hypothesis $H_1 : \lambda(\cdot) \notin \mathcal{C}$, where the functional form of $\lambda_0(\cdot; \xi)$ is known, except for the $p \times 1$ vector ξ. The parameter ξ is a nuisance parameter in this testing problem.

Following Peña (1998b), let $\lambda(\cdot)$ be the true, but unknown hazard function, and consider the class of functions $\mathcal{K} = \{\kappa(\cdot; \xi) : \xi \in \Xi\}$ where

$$\kappa(t; \xi) = \log\left\{\frac{\lambda(t)}{\lambda_0(t; \xi)}\right\}.$$

If the hypothesized class $\mathcal{C}$ is a reasonable one, then the functional class $\mathcal{K}$ will possess certain properties such as being a Hilbert space. We shall therefore assume that there exists a basis set of functions $\{(\psi_1(t; \xi), \psi_2(t; \xi), \ldots) : t \in \mathcal{T}\}$, which maybe composed of trigonometric, polynomial, or wavelet functions. Therefore,

$$\kappa(t;\xi) = \log\left\{\frac{\lambda(t)}{\lambda_0(t;\xi)}\right\} = \sum_{i=1}^{\infty} \theta_i \psi_i(t;\xi),\ t \in \mathcal{T}. \tag{1.1}$$

By Parseval's Theorem, coefficients in the later terms of the right hand side of (1.1) will be small, so a reasonable approximation can be obtained by truncating the infinite sum at some point k, called a *smoothing* order. Such a truncation was referred to as a 'Neyman truncation' by Fan (1996). We will partially address the choice of the value of k in Section 4. Thus, (1.1) may be expressed as

$$\kappa(t;\xi) = \log\left\{\frac{\lambda(t)}{\lambda_0(t;\xi)}\right\} \approx \sum_{i=1}^{k} \theta_i \psi_i(t;\xi) = \boldsymbol{\theta}'\boldsymbol{\psi}(t;\xi),\ t \in \mathcal{T},$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ are $k \times 1$ vectors. This results in an embedding class for the hazard rate functions given by

$$\mathcal{A}_k = \{\lambda_k(\cdot;\boldsymbol{\theta},\xi,\beta) = \lambda_0(\cdot;\xi)\exp[\boldsymbol{\theta}'\boldsymbol{\psi}(\cdot;\xi)] : \boldsymbol{\theta} \in \mathbb{R}^k\}. \tag{1.2}$$

Note that $\mathcal{C} \subset \mathcal{A}_k$ since it obtains by setting $\boldsymbol{\theta} = \mathbf{0}$. Consequently, within this embedding, we could test for the composite hypotheses

$$H_0^* : (\boldsymbol{\theta},\xi,\beta) \in \{\mathbf{0}\} \times \Xi \times \mathcal{B} \quad \text{versus} \quad H_1^* : (\boldsymbol{\theta},\xi,\beta) \in \Re^k \setminus \{\mathbf{0}\} \times \Xi \times \mathcal{B}.$$

In this formulation, the proposed class of tests will be the score tests for the above hypotheses. Such tests will possess certain local optimality properties intrinsic to score tests (cf., Cox and Hinkley, 1974; Choi, et al., 1996). Another desirable property of this class of tests is its power to detect a wide range of alternatives. See, for instance, Rayner and Best (1989) for the classical density-based formulation, and Peña (1998ab) and Agustin and Peña (2000) for the hazard-based formulation. In addition, both omnibus and directional tests can be generated from this class of tests. Various empirical studies such as Kopecky and Pierce (1979), Miller and Quesenberry (1979), and Eubank and LaRiccia (1992) have shown that smooth tests possess more power than commonly used omnibus test statistics for a larger class of feasible alternatives. In particular, the

Cramer-von Mises (CVM) statistic has been shown to have dismal performance against almost all but location-scale alternatives. The paper by Eubank and LaRiccia (1992) presents theoretical justification for the observed phenomenon that smooth tests have superior power over CVM type statistics for many non-location-scale alternatives. Furthermore, a natural consequence of this hazard-based formulation is our ability to obtain goodness-of-fit tests based on the model's generalized residuals, which are usually utilized for validation purposes. For instance, we are able to generalize existing tests based on martingale residuals. The hazard-based approach also afforded us the machinery to extend our results to the recurrent event setting. Since situations where the event of interest may recur are encountered in a variety of disciplines, it is important to develop statistical inference procedures for stochastic models dealing with recurrent data. The particular application that we deal with involves goodness-of-fit problems for the distribution of the time-to-first-event-occurrence in a general recurrent event model, which is the Block, Borges and Savits (1985) minimal repair model but incorporating covariates.

A major difference between the settings considered in earlier papers and this paper is the need to contend with two sets of nuisance parameters: ξ and β. Of interest is the ascertainment of the effects of estimating these nuisance parameters. This issue is especially important when using generalized residuals where unknown parameters are replaced by their estimates. Results presented later show that the use of estimates in lieu of the unknown parameters warrants adjustments on the asymptotic variances of the test statistics, and by not doing so, erroneous conclusions could arise since the seemingly intuitive approach of using plug-in estimates for unknown parameters could have drastic consequences, even when the plug-in estimators are consistent. This issue has been examined also in simpler settings, such as in Pierce (1982), Randles (1982, 1984), Lagakos (1981), Baltazar-Aban and Peña (1995), and Peña (1995, 1998ab).

We now outline the organization of this paper. In Section 2, we present the proposed class of goodness-of-fit tests and indicate its development. The asymptotic properties of the test statistics under a contiguous sequence of local alternatives is provided in subsection 2.2, although to make the paper more readable and to conserve space, we omit the technical proofs and simply direct the reader to the relevant references. The required technicality conditions for the asymptotic properties are however enumerated in an appendix. An application of specific members of the class of tests to a generalized recurrent event model is performed in Section 3 where closed-form expressions for test statistics are obtained in the case of time-independent and Bernoulli-distributed covariates. In Section 4 simulation results pertaining to the finite sample size properties of the tests are presented. Finally, Section 5 provides illustrative examples which also serve as a venue for discussing issues such as censoring and other covariate structures.

2. Class of GOF Tests

2.1 Score Process

Within the embedding given in (1.2), the relevant score process for $\boldsymbol{\theta}$ is

$$\boldsymbol{U}^F_{\boldsymbol{\theta}}(t;\boldsymbol{\theta},\xi,\beta) = \sum_{j=1}^{n}\int_0^t \boldsymbol{\psi}(s;\xi)\mathrm{d}M_j(s;\boldsymbol{\theta},\xi,\beta), \tag{2.1}$$

where $M_j(s;\boldsymbol{\theta},\xi,\beta) = N_j(s) - A_j(s;\boldsymbol{\theta},\xi,\beta)$ and

$$A_j(t;\boldsymbol{\theta},\xi,\beta) = \int_0^t Y_j(s)\lambda_0(s;\xi)\exp\{\beta'\boldsymbol{X}_j(s)\}\exp\{\boldsymbol{\theta}'\boldsymbol{\psi}(s;\xi)\}\ \mathrm{d}s.$$

Suppose (ξ_0,β_0) is the true, but unknown, value of (ξ,β). Under H_0^*,

$$\boldsymbol{M}(\cdot) = (M_1(\cdot;\mathbf{0},\xi_0,\beta_0),\ldots,M_n(\cdot;\mathbf{0},\xi_0,\beta_0))$$

is a vector of local square-integrable orthogonal martingales. Hence, since $\boldsymbol{\psi}$ is predictable, the score process evaluated at $(\boldsymbol{\theta},\xi,\beta) = (\mathbf{0},\xi_0,\beta_0)$ given by

$$\boldsymbol{U}^F_{\boldsymbol{\theta}}(t;\mathbf{0},\xi_0,\beta_0) = \sum_{j=1}^{n}\int_0^t \boldsymbol{\psi}(s;\xi_0)\ \mathrm{d}M_j(s;\mathbf{0},\xi_0,\beta_0) \tag{2.2}$$

is a local square-integrable martingale. However, the process in (2.2) is not observable since (ξ_0,β_0) is unknown, so we plug-in estimators for these unknown nuisance parameters.

For the CPHM, it is usual to use the partial likelihood maximum likelihood estimator (PLMLE) (cf., Keiding, et al., 1998) of β, denoted by $\hat{\boldsymbol{\beta}}$, which is a solution of the equation $\boldsymbol{U}_{\beta}^{P}(\tau;\xi,\beta)=\mathbf{0}$, where

$$\boldsymbol{U}_{\beta}^{P}(\tau;\xi,\beta)=\sum_{j=1}^{n}\int_{0}^{\tau}\left[\boldsymbol{X}_{j}(s)-\boldsymbol{E}(s;\beta)\right]\ \mathrm{d}N_{j}(s);$$

$$\boldsymbol{S}_{(m)}(t;\beta)=\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{X}_{j}(t)^{\otimes m}Y_{j}(t)\exp[\beta'\boldsymbol{X}_{j}(t)],\quad m=0,1,2;$$

$$\boldsymbol{E}(t;\beta)=\frac{\boldsymbol{S}_{(1)}(t;\beta)}{S_{(0)}(t;\beta)},$$

and for a vector $\boldsymbol{a}$, $\boldsymbol{a}^{\otimes 0}=1$, $\boldsymbol{a}^{\otimes 1}=\boldsymbol{a}$, and $\boldsymbol{a}^{\otimes 2}=\boldsymbol{a}\boldsymbol{a}'$. To estimate ξ, we use the estimator that maximizes the profile likelihood. This estimator of ξ, denoted by $\hat{\xi}\equiv\hat{\xi}(\hat{\beta})$, solves the equation

$$\boldsymbol{U}_{\xi}^{F}(\tau;\xi,\hat{\beta})\equiv\sum_{j=1}^{n}\int_{0}^{\tau}\rho(s;\xi)\ \mathrm{d}M_{j}(s;\mathbf{0},\xi,\hat{\beta})=\mathbf{0},$$

where $\rho(s;\xi)=\nabla_{\xi}\log\lambda_{0}(s;\xi)$ with $\nabla_{\xi}=\partial/\partial\xi$ being the gradient operator for ξ. Consequently, the vector of score statistics for testing H_0^* is

$$\boldsymbol{U}_{\boldsymbol{\theta}}^{F}(\tau;\mathbf{0},\hat{\xi},\hat{\beta})=\sum_{j=1}^{n}\int_{0}^{\tau}\boldsymbol{\psi}(s;\hat{\xi})\ \mathrm{d}M_{j}(s;\mathbf{0},\hat{\xi},\hat{\beta}).\tag{2.3}$$

Since (ξ,β) was replaced by $(\hat{\xi},\hat{\beta})$, $M_j(s;\mathbf{0},\hat{\xi},\hat{\beta})$'s are no longer martingales. In order to construct the testing procedure and to assess the effects of plugging-in estimators for the unknown nuisance parameters, we need the sampling distribution of $\boldsymbol{U}_{\boldsymbol{\theta}}^{F}(\tau;\mathbf{0},\hat{\xi},\hat{\beta})$. However, the exact sampling distribution of this estimated score process is not analytically tractable, so we focus on its asymptotic properties under a contiguous sequence of local alternatives.

2.2 Asymptotics

Consider a sequence of models indexed by n with processes $\{(N_j^{(n)},Y_j^{(n)},\boldsymbol{X}_j^{(n)}),\ j=1,\ldots,n\}$ defined on probability spaces $(\Omega^{(n)},\mathcal{F}^{(n)},\mathcal{P}^{(n)})$ and adapted to the filtrations $\boldsymbol{F}^{(n)}=\{\mathcal{F}_t^{(n)}:t\in\mathcal{T}\}$. The sequence of compensators is $\{A_j^{(n)}(\cdot;\boldsymbol{\theta}^{(n)},\xi,\beta):j=1,\ldots,n;n=1,2,\ldots\}$ with

$$A_j^{(n)}(t;\boldsymbol{\theta}^{(n)},\xi,\beta)=\int_0^t Y_j^{(n)}(s)\lambda_0(s;\xi)\exp\{\boldsymbol{\theta}^{(n)\prime}\boldsymbol{\psi}^{(n)}(s;\xi)\}\exp\{\beta'\boldsymbol{X}_j^{(n)}(s)\}\ \mathrm{d}s.$$

Notice that k, $\lambda_0(s;\xi)$, and β are independent of n. The sequence of hypotheses that are of interest is

$$\begin{aligned} H_0^{(n)} : (\boldsymbol{\theta}^{(n)}, \xi, \beta) &\in \{\mathbf{0}\} \times \Xi \times \mathcal{B}; \\ H_1^{(n)} : (\boldsymbol{\theta}^{(n)}, \xi, \beta) &\in \left\{\frac{\boldsymbol{\gamma}}{\sqrt{n}}(1+o(1))\right\} \times \Xi \times \mathcal{B}, \end{aligned}$$

where $\gamma \in \mathbb{R}^k$. Therefore, the sequence of score processes is given by

$$\boldsymbol{U}_\theta^{(n)}(t; \mathbf{0}, \hat{\xi}^{(n)}, \hat{\beta}^{(n)}) = \sum_{j=1}^{n} \int_0^t \boldsymbol{\psi}^{(n)}(s; \hat{\xi}^{(n)})\, \mathrm{d}M_j^{(n)}(s; \mathbf{0}, \hat{\xi}^{(n)}, \hat{\beta}^{(n)}),$$

where $M_j^{(n)}(t; \mathbf{0}, \xi, \beta) = N_j^{(n)}(t) - \int_0^t Y_j^{(n)}(s)\lambda_0(s;\xi)\exp\{\beta' \boldsymbol{X}_j^{(n)}(s)\}\,\mathrm{d}s$ and $\hat{\xi}^{(n)} \equiv \hat{\xi}^{(n)}(\hat{\beta}^{(n)})$. Henceforth, we suppress the superscript n. We introduce the processes

$$\boldsymbol{P}_{(m)}(t; \xi, \beta) = \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{\psi}(t;\xi)^{\otimes m} Y_j(t) \exp\{\beta' \boldsymbol{X}_j(t)\}, \quad m = 1, 2.$$

Moreover, since the ensuing discussion focuses on the results at time τ, we further simplify our notation by suppressing the time argument. We denote by lower case letters in-probability limiting functions of the processes denoted by the corresponding capital letters. For instance, $s_{(1)}$ is the in-probability limit of $S_{(1)}$. Letting

$$\boldsymbol{e} = \frac{\boldsymbol{s}_{(1)}}{s_{(0)}}, \quad \boldsymbol{v} = \frac{\boldsymbol{s}_{(2)}}{s_{(0)}} - \boldsymbol{e}^{\otimes 2}, \quad \text{and} \quad \boldsymbol{d}_{(m)} = \frac{\boldsymbol{p}_{(m)}}{s_{(0)}}, \quad (m = 1, 2),$$

we define the following matrices, where for brevity of notation, we have taken the liberty of suppressing the time argument s, which is the variable of integration, in the functions/processes of the integrands:

$$\begin{aligned} \boldsymbol{\Sigma}_{11}(\xi, \beta) &= \int_0^\tau \boldsymbol{d}_{(2)}(\xi, \beta) s_{(0)}(\beta) \lambda_0(\xi)\ \mathrm{d}s, \\ \boldsymbol{\Sigma}_{12}(\xi, \beta) &= \int_0^\tau \boldsymbol{d}_{(1)}(\xi, \beta) \rho(\xi)' s_{(0)}(\beta) \lambda_0(\xi)\ \mathrm{d}s, \\ \boldsymbol{\Sigma}_{22}(\xi, \beta) &= \int_0^\tau \rho(\xi)^{\otimes 2} s_{(0)}(\beta) \lambda_0(\xi)\ \mathrm{d}s, \\ \boldsymbol{\Sigma}_{33}(\xi, \beta) &= \int_0^\tau \boldsymbol{v}(\beta) s_{(0)}(\beta) \lambda_0(\xi) \mathrm{d}s, \\ \boldsymbol{\Delta}_1(\xi, \beta) &= \int_0^\tau \boldsymbol{d}_{(1)}(\xi, \beta) \boldsymbol{e}(\beta)' s_{(0)}(\beta) \lambda_0(\xi)\ \mathrm{d}s, \\ \boldsymbol{\Delta}_2(\xi, \beta) &= \int_0^\tau \rho(\xi) \boldsymbol{e}(\beta)' s_{(0)}(\beta) \lambda_0(\xi)\ \mathrm{d}s, \end{aligned}$$

and

$$\boldsymbol{\Sigma}(\xi_0,\beta_0) = \begin{bmatrix} \boldsymbol{\Sigma}_{11}(\xi_0,\beta_0) & \boldsymbol{\Sigma}_{12}(\xi_0,\beta_0) & \mathbf{0} \\ \boldsymbol{\Sigma}_{21}(\xi_0,\beta_0) & \boldsymbol{\Sigma}_{22}(\xi_0,\beta_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_{33}(\xi_0,\beta_0) \end{bmatrix}.$$

To facilitate economy of notation in the sequel, we also define the following matrices:

$$\begin{aligned} \boldsymbol{\Sigma}_{11.2}(\xi,\beta) &= \boldsymbol{\Sigma}_{11}(\xi,\beta) - \boldsymbol{\Sigma}_{12}(\xi,\beta)\boldsymbol{\Sigma}_{22}(\xi,\beta)^{-1}\boldsymbol{\Sigma}_{21}(\xi,\beta); \\ \boldsymbol{\Upsilon}(\xi,\beta) &= \boldsymbol{\Delta}_1(\xi,\beta) - \boldsymbol{\Sigma}_{12}(\xi,\beta)\boldsymbol{\Sigma}_{22}(\xi,\beta)^{-1}\boldsymbol{\Delta}_2(\xi,\beta). \end{aligned}$$

To state the asymptotic results, certain regularity conditions are needed. These conditions are a combination of those used in Peña (1998ab) or some variant thereof. The list of conditions brings together those in Andersen and Gill (1982), Borgan (1984), and conditions regulating behavior of the $\boldsymbol{\psi}_j$ processes. We enumerate these conditions in the Appendix. So as to conserve space and at the same time avoid distracting the reader from the main emphasis of this paper, which is the goodness-of-fit procedure and the examination of the impact of estimating unknown parameters, we omitted the presentation of all intermediate results and proofs. Instead, we refer the interested reader to technical details in Peña (1998b) and Peña and Agustin (2001). The main asymptotic result needed in the test construction is as follows.

Theorem 2.1: *If conditions (I) – (XII) in the Appendix hold, then under a sequence of contiguous local alternatives* $H_1^{(n)} : (\boldsymbol{\theta},\xi,\beta) \in \left\{\frac{\boldsymbol{\gamma}}{\sqrt{n}}(1+o(1))\right\} \times \Xi \times \mathcal{B}$*, the estimated score process* $\frac{1}{\sqrt{n}}\boldsymbol{U}_\theta^F(\mathbf{0},\hat{\xi},\hat{\beta})$ *converges weakly on Skorokhod's* $D(\mathcal{T})^k$ *space to* $\tilde{\boldsymbol{Z}}(\xi_0,\beta_0)$*, a Gaussian process with mean function* $\boldsymbol{\mu}(\xi_0,\beta_0) = \boldsymbol{\Sigma}_{11.2}(\xi_0,\beta_0)\boldsymbol{\gamma}$ *and covariance matrix function*

$$\boldsymbol{\Gamma}(\xi_0,\beta_0) = \boldsymbol{\Sigma}_{11.2}(\xi_0,\beta_0) + \boldsymbol{\Upsilon}(\xi_0,\beta_0)\boldsymbol{\Sigma}_{33}(\xi_0,\beta_0)^{-1}\boldsymbol{\Upsilon}(\xi_0,\beta_0)'.$$

¿From Theorem 2.1, it follows that the quantity

$$S(\xi_0,\beta_0) = \frac{1}{n}\left\{\boldsymbol{U}_\theta^F(\hat{\xi},\hat{\beta})'\right\}\{\boldsymbol{\Gamma}(\xi_0,\beta_0)\}^{-}\left\{\boldsymbol{U}_\theta^F(\hat{\xi},\hat{\beta})\right\},$$

where $\boldsymbol{\Gamma}^{-}(\cdot)$ is a generalized inverse of $\boldsymbol{\Gamma}(\cdot)$, converges in distribution, under $H_1^{(n)}$, to a noncentral chi-square distribution with degrees of freedom $k^* = \text{rank}\,[\boldsymbol{\Gamma}(\xi_0,\beta_0)]$ and noncentrality parameter $\delta^2_{p(\cdot)} = \boldsymbol{\mu}'(\xi_0,\beta_0)\boldsymbol{\Gamma}^{-}(\xi_0,\beta_0)\boldsymbol{\mu}(\xi_0,\beta_0)$. Note that $S(\xi_0,\beta_0)$ is not a statistic since $\boldsymbol{\Gamma}^{-}(\xi_0,\beta_0)$

depends on the unknown parameter vector (ξ_0, β_0). A possible consistent estimator of $\mathbf{\Sigma}(\xi_0, \beta_0)$ is given by

$$\hat{\mathbf{\Sigma}}(\hat{\xi}, \hat{\beta}) = \frac{1}{2n} \sum_{j=1}^{n} \int_0^\tau \begin{bmatrix} \boldsymbol{\psi}(\hat{\xi}) \\ \rho(\hat{\xi}) \\ \boldsymbol{X}_j - \mathbf{E}(\hat{\beta}) \end{bmatrix}^{\otimes 2} \left\{ \mathrm{d}N_j + Y_j \lambda_0(\hat{\xi}) \exp\{\hat{\beta}' \boldsymbol{X}_j\} \, \mathrm{d}s \right\}, \tag{2.4}$$

a convex combination of the estimators based on the predictable variation and optional variation processes. Also, $\mathbf{\Delta}_1(\xi, \beta)$ and $\mathbf{\Delta}_2(\xi, \beta)$ could be consistently estimated by

$$\hat{\mathbf{\Delta}}_1 = \int_0^\tau \boldsymbol{P}_{(1)}(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\beta}}) \boldsymbol{E}(\hat{\boldsymbol{\beta}})' \lambda_0(\hat{\boldsymbol{\xi}}) \mathrm{d}s \quad \text{and} \quad \hat{\mathbf{\Delta}}_2 = \int_0^\tau \boldsymbol{\rho}(\hat{\boldsymbol{\xi}}) \boldsymbol{E}(\hat{\boldsymbol{\beta}})' S_{(0)}(\hat{\boldsymbol{\beta}}) \lambda_0(\hat{\boldsymbol{\xi}}) \mathrm{d}s.$$

An estimator of $\mathbf{\Gamma}(\boldsymbol{\xi}, \boldsymbol{\beta})$, which we denote by $\hat{\mathbf{\Gamma}}(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\beta}})$, is then immediately formed from the aforementioned matrices. Furthermore, a consistent estimator of k^* is $\hat{k}^* = \text{rank}\ [\hat{\mathbf{\Gamma}}(\hat{\xi}, \hat{\beta})]$ (cf., Theorem 2, Li and Doss, 1993).

The asymptotic distribution under H_0 is obtained by setting $\gamma = \mathbf{0}$ in Theorem 2.1. Doing so, it follows that an asymptotic α-level goodness-of-fit test for $H_0 : \lambda(\cdot) \in \mathcal{C} \equiv \{\lambda_0(\cdot; \xi) : \xi \in \Xi\}$ rejects H_0 in favor of $H_1 : \lambda(\cdot) \notin \mathcal{C}$ whenever

$$S(\hat{\xi}, \hat{\beta}) = \frac{1}{n} \boldsymbol{U}_\theta^F(\hat{\xi}, \hat{\beta})' \hat{\mathbf{\Gamma}}^-(\hat{\xi}, \hat{\beta}) \boldsymbol{U}_\theta^F(\hat{\xi}, \hat{\beta}) \geq \chi^2_{\hat{k}^*;\alpha}, \tag{2.5}$$

where $\chi^2_{\hat{k}^*;\alpha}$ is the $(1-\alpha)100^{\text{th}}$ percentile of the central chi-square distribution with $\hat{k}^*$ degrees of freedom. ¿From Theorem 2.1 it also follows that the asymptotic local power (ALP) of the test described above for direction vector $\boldsymbol{\gamma}$ is

$$\text{ALP}(\boldsymbol{\gamma}) = \mathbf{P} \left\{ \chi^2_{k^*}[\delta^2(\boldsymbol{\gamma})] \geq \chi^2_{k^*;\alpha} \right\}, \tag{2.6}$$

where $\chi^2_{k^*}[\delta^2]$ is a non-central chi-square random variable with k^* degrees-of-freedom and non-centrality parameter δ^2. For the case at hand, the noncentrality parameter is given by

$$\delta^2(\boldsymbol{\gamma}) = \boldsymbol{\gamma}' \mathbf{\Sigma}_{11.2}(\boldsymbol{\xi}_0, \boldsymbol{\beta}_0) \{\mathbf{\Gamma}(\boldsymbol{\xi}_0, \boldsymbol{\beta}_0)\}^- \mathbf{\Sigma}_{11.2}(\boldsymbol{\xi}_0, \boldsymbol{\beta}_0) \boldsymbol{\gamma}.$$

2.3 Effects of Plug-In Procedure

We discuss in this subsection the effects of plugging the estimators $(\hat{\xi}, \hat{\beta})$ for the unknown parameters (ξ, β). First, note that if the true values of (ξ_0, β_0) of (ξ, β) are known, then the covariance function is simply $\mathbf{\Sigma}_{11}(\xi_0, \beta_0)$. Hence, from the covariance matrix in Theorem 2.1, the plug-in approach has *no* asymptotic effect whenever $\mathbf{\Sigma}_{12}(\xi_0, \beta_0)$ and $\mathbf{\Delta}_1(\xi_0, \beta_0)$ are both $\mathbf{0}$. These conditions are orthogonality conditions between $\boldsymbol{\psi}(\xi_0, \beta_0)$ and $\rho(\xi_0)$ and between $\boldsymbol{\psi}(\xi_0, \beta_0)$ and $e(\beta_0)$, respectively. In the event that these conditions are not met, this plug-in approach entails adjustments in the covariance function. Such adjustments need to be taken into account to come up with an appropriate test. Ignoring these adjustments could lead to erroneous conclusions. In the setting with i.i.d. observations and when the restricted estimators are obtained by maximizing the full likelihood with $\boldsymbol{\theta} = \mathbf{0}$, the covariance function is constituted only by the term $\mathbf{\Sigma}_{11.2}$. Thus, the term $\mathbf{\Upsilon}\mathbf{\Sigma}_{33}^{-1}\mathbf{\Upsilon}'$ can be attributed to the use of the partial likelihood to obtain the PLMLE of β, and this PLMLE was subsequently used in the profile likelihood to obtain the estimator of ξ. The use of the less efficient PLMLE for estimating β rather than the restricted MLE resulted in an increase in the variance. We opted to use the PLMLE because this is typically the estimator that is used in practice (cf., Keiding, et al., 1998) for Cox-type models owing to the availability of softwares that enables its computation and because the computation of the restricted MLE will usually be more involved.

In the next section, we illustrate the magnitude of the necessary adjustments for a generalized recurrent event model. Notice that if we consider the case where the class $\mathcal{C}$ is fully specified, then the covariance matrix simplifies to

$$\mathbf{\Gamma}(\xi_0, \beta_0) = \mathbf{\Sigma}_{11}(\xi_0, \beta_0) + \mathbf{\Delta}_1(\xi_0, \beta_0)\mathbf{\Sigma}_{33}^{-1}(\xi_0, \beta_0)\mathbf{\Delta}_1(\xi_0, \beta_0)'.$$

This is precisely the result in Peña (1998a) for the simple hypothesis case. On the other hand, if we consider the no-covariate case by setting $\boldsymbol{X}_j(s) = \mathbf{0}$, $j = 1, 2, \ldots, n$, then the covariance matrix reduces to

$$\mathbf{\Gamma}(\xi_0, \beta_0) = \mathbf{\Sigma}_{11}(\xi_0, \beta_0) - \mathbf{\Sigma}_{12}(\xi_0, \beta_0)\mathbf{\Sigma}_{22}^{-1}(\xi_0, \beta_0)\mathbf{\Sigma}_{21}(\xi_0, \beta_0).$$

Hence, we recover the result in Peña (1998b) for the no-covariate composite hypothesis setting as well. As such, the results in the present paper generalizes existing results.

3. Application to a Generalized Recurrent Event Model

The analysis of models pertaining to recurrent events arises in various fields of study such as biomedical, engineering, economics, sociological, and financial settings. In this section, though we will be using reliability terminology and associate recurrent events with failures or repairs in repairable systems, the model and inferential methods that will be discussed are also relevant and applicable in other settings. We focus on the Block, Borges and Savits (1985) minimal repair model, hereon referred to as the BBS model. We briefly describe this model. Consider a component that is placed on test at time 0. If the component fails at time t, either a perfect repair is done with probability $p(t)$ or a minimal repair is undertaken with probability $q(t) = 1 - p(t)$. A perfect repair restores the component to the good-as-new state, that is, its effective age reverts to 0; whereas, a minimal repair restores the effective age to that just before the failure. In other words, after a minimal repair, the distribution of the time to the next failure is stochastically equivalent to that of a working system of the same effective age. Without loss of generality, we assume that repairs take negligible time. This process of perfectly or minimally repairing the component takes place at each subsequent failure with the probability associated with the type of repair dependent on the effective age of the system. Though the BBS model is primarily utilized in the reliability and operations research settings, it is also applicable to other areas since it admits as special cases some of the models commonly encountered in practice. For instance, if $p(t) = 1$, we recover the independent and identically distributed (i.i.d.) model, also called the renewal model. A common model used for recurrent events in the biomedical setting is the nonhomogeneous Poisson process. This model is a special case of the BBS model obtained by taking $p(t) = 0$.

We consider a generalized BBS model which admits covariates. Denote by $W_0 \equiv 0 < W_1 < W_2 < \ldots$ the successive failure times of a component, and let $U_1, U_2, \ldots$ be a sequence of i.i.d. Uniform[0,1] random variables which are independent of the failure times. The sequence $(W_1, W_2, \ldots, W_\nu)$, where $\nu = \inf\{k \in \{1, 2, \ldots\} : U_k < p(W_k)\}$, is an epoch of the BBS model. Consider observing n independent BBS epochs $\{W_{jk} : 1 \leq j \leq n, 1 \leq k \leq \nu_j\}$ associated with n units where the j^{th} unit has a possibly time-dependent covariate process $\boldsymbol{X}_j(\cdot)$. Define the stochastic processes $\boldsymbol{N} = \{N_1(t), N_2(t), \ldots, N_n(t)\}$ and $\boldsymbol{Y} = \{Y_1(t), Y_2(t), \ldots, Y_n(t)\}$, where

$$N_j(t) = \sum_{k=1}^{\infty} I\{W_{jk} \leq t \wedge W_{j\nu_j}\} \quad \text{and} \quad Y_j(t) = I\{W_{j\nu_j} \geq t\}.$$

With respect to the filtration $\boldsymbol{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$, where $\mathcal{F}_t = \mathcal{F}_0 \vee \bigvee_{j=1}^n \sigma\{(N_j(s), Y_j(s)) : s \leq t\}$, and with $\mathcal{F}_0$ containing all information available at time 0, the compensator of $\boldsymbol{N}$ is $\boldsymbol{A} = \{(A_1(t; \xi, \beta), \ldots, A_n(t; \xi, \beta)) : t \in \mathcal{T}\}$ with

$$A_j(t; \xi, \beta) = \int_0^t Y_j(s) \lambda(s) \exp\{\beta' \boldsymbol{X}_j(s)\} \, \mathrm{d}s,$$

where $\lambda(s)$ is some baseline hazard function, β is a $q \times 1$ vector of regression coefficients, and $\boldsymbol{X}_1(s), \ldots, \boldsymbol{X}_n(s)$ are $q \times 1$ vectors of locally bounded predictable covariate processes.

In the sequel, the following condition is assumed to hold:

$$\int_0^\infty p(t) \lambda(t) \exp\{\beta' \boldsymbol{X}_j(t)\} \, \mathrm{d}t = \infty. \tag{3.1}$$

Condition (3.1) guarantees that the waiting time to the first perfect repair is almost surely finite with hazard rate function $\lambda^*(t) = p(s)\lambda(s)\exp\{\beta' \boldsymbol{X}_j(s)\}$ (c.f., Block, et al., 1985). With this generalized BBS model, we present an example to illustrate the magnitude of the required adjustments due to the plug-in procedure.

Example 3.1: Suppose $k = 1, p(t) = p$, and the hypothesized class of baseline hazard functions consists of $\Lambda_0(t; \xi) = \xi t I\{t \geq 0\}$ corresponding to constant failure rates. The assumption that

$p(t) = p$ reduces the BBS model to the Brown and Proschan (1983) imperfect repair model. Moreover, suppose that the covariate X is time-independent and Bernoulli distributed with success parameter p^*. Then

$$
\begin{aligned}
\boldsymbol{\Sigma}_{11}(\xi_0,\beta_0) &= \xi_0 \int_0^\tau \psi^2(s;\xi_0)\left\{(1-p^*)\exp[-p\xi_0 s] \ + p^*\exp[\beta_0 - p\xi_0\exp(\beta_0)s]\right\}\mathrm{d}s;\\
\boldsymbol{\Sigma}_{12}(\xi_0,\beta_0) &= \int_0^\tau \psi(s;\xi_0)\left\{(1-p^*)\exp[-p\xi_0 s] \ + p^*\exp[\beta_0 - p\xi_0\exp(\beta_0)s]\right\}\mathrm{d}s;\\
\boldsymbol{\Sigma}_{22}(\xi_0,\beta_0) &= \frac{1}{p\xi_0^2}\left\{(1-p^*)[1-\exp\{-p\xi_0\tau\}] \ + p^*[1-\exp\{-p\xi_0\exp(\beta_0)\tau\}]\right\};\\
\boldsymbol{\Delta}_{1}(\xi_0,\beta_0) &= p^*\xi_0\int_0^\tau \psi(s;\xi_0)\exp[\beta_0 - p\xi_0\exp(\beta_0)s]\ \mathrm{d}s;\\
\boldsymbol{\Delta}_{2}(\xi_0,\beta_0) &= \frac{p^*}{p\xi_0}\left[1-\exp\{-p\xi_0\exp(\beta_0)\tau\}\right];\\
\boldsymbol{\Sigma}_{33}(\xi_0,\beta_0) &= p^*(1-p^*)\xi_0\exp(\beta_0)\int_0^\tau \frac{\exp[-p\xi_0\exp(\beta_0)s]}{(1-p^*)+p^*\exp(\beta_0)\exp[-p\xi_0\exp(\beta_0-1)s]}\ \mathrm{d}s.
\end{aligned}
$$

If, furthermore, we focus our attention to the case where $\beta_0 = 0$ and $\psi(t) = \exp[-\Lambda_0(t)] = \exp[-\xi_0 t]$, then the above expressions simplify to

$$
\begin{aligned}
\Sigma_{11}(\xi_0,0) &= \frac{1-\exp[-(p+2)\xi_0\tau]}{p+2};\\
\Sigma_{12}(\xi_0,0) &= \frac{1-\exp[-(p+1)\xi_0\tau]}{\xi_0(p+1)};\\
\Sigma_{22}(\xi_0,0) &= \frac{1-\exp[-p\xi_0\tau]}{p\xi_0^2};\\
\Delta_{1}(\xi_0,0) &= \frac{p^*}{1+p}[1-\exp\{-(p+1)\xi_0\tau\}];\\
\Delta_{2}(\xi_0,0) &= \frac{p^*}{p\xi_0}\left[1-\exp\{-p\xi_0\tau\}\right];\\
\Sigma_{33}(\xi_0,0) &= \frac{p^*(1-p^*)}{p}[1-\exp\{-p\xi_0\tau\}].
\end{aligned}
$$

If we let $\tau \to \infty$, which translates to letting our observation period cover a complete epoch for each component, then

$$\Gamma(\xi_0,0) = \frac{1}{p+2} - \frac{p}{(p+1)^2}.$$

Taking $p = \frac{1}{2}$ yields $\Sigma_{11}(\xi_0,0) = \frac{2}{5}$ and $\Gamma(\xi_0,0) = \frac{8}{45}$. Hence, the adjustment resulted in a 56% decrease in the variance! Lest the reader get the impression that the applicability of our results is limited to the simple scenario considered in the preceding example, we emphasize that the use of the simple covariate structure was for the purpose of obtaining closed form expressions in order to vividly illustrate the effects of the plug-in procedure.

The discussion that now follows applies to the CPHM involving a BBS model with a general covariate structure. A key element of the proposed class of tests is the process $\psi(\cdot)$ since it characterizes the family of alternatives that a particular test will detect powerfully. As previously mentioned, $\psi(\cdot)$ could be a trigonometric, polynomial, or wavelet basis set, and furthermore, our formulation allows the possibility that the basis set is random. In keeping with the spirit of Neyman's (1937) smooth goodness-of-fit tests, we focus on a polynomial specification for $\psi(\cdot)$. Aside from the fact that polynomials form a basis for most functions, the tests resulting from a polynomial specification assume fairly simple forms which makes them appealing to use in practice, and furthermore, the resulting test statistics are simply functions of the model's generalized residuals. This polynomial specification also generates omnibus and directional tests with reasonable powers to detect a wide range of alternatives even relative to other choices. This aspect will be discussed in the next section when we present the simulation results.

Let us consider the "polynomial" specification

$$\boldsymbol{\psi}(t : \mathrm{PW}_k) = [\Lambda_0(t;\xi), \Lambda_0^2(t : \xi), \ldots, \Lambda_0^k(t : \xi)]',$$

where $k \in \{1, 2, \ldots\}$ is some fixed smoothing order. Note that this polynomial form is slightly different from the one in Peña (1998a) and Agustin and Peña (2000). To obtain closed-form expressions, we assume time-independent covariates. In the event that one or more of the covariates vary with time, the corresponding expressions can be derived directly from (2.3) and (2.4). In the case of time-independent covariates, the score statistic resulting from this specification is

$$\boldsymbol{Q}(\mathrm{PW}_k) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[\sum_{i=1}^{N_j(\tau)} R_{ji}^{\ell} - \exp\{\hat{\beta}' \boldsymbol{X}_j\} \frac{(R_{j\nu_j}^{\tau})^{\ell+1}}{\ell+1} \right]_{\ell=1,\ldots,k},$$

where

$$R_{ji} = \Lambda_0(W_{ji}; \hat{\xi}), \quad (i = 1, \ldots, \nu_j, j = 1, \ldots, n) \quad \text{and} \quad R_{j\nu_j}^{\tau} = \Lambda_0(\tau \wedge W_{j\nu_j}; \hat{\xi})$$

are the model's generalized residuals. Moreover, the components of the estimated covariance matrix are found to be

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}}_{11}(\mathrm{PW}_k) &= \frac{1}{2n} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{N_j(\tau)} \left[R_{ji}^{\ell+\ell'} \right]_{\ell,\ell'=1,\ldots,k} + \exp\{\hat{\beta}' \boldsymbol{X}_j\} \left[\left(\frac{(R_{j\nu_j}^{\tau})^{\ell+\ell'+1}}{\ell+\ell'+1} \right)_{\ell,\ell'=1,\ldots,k} \right] \right\}; \\
\hat{\boldsymbol{\Sigma}}_{12}(\mathrm{PW}_k) &= \frac{1}{2n} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{N_j(\tau)} [R_{ji}^{\ell}]_{\ell=1,\ldots,k} \rho(W_{ji}; \hat{\xi})' + \exp\{\hat{\beta}' \boldsymbol{X}_j\} \times \right. \\
&\qquad \left. \int_0^{\tau} \left[\Lambda_0^{\ell}(s; \hat{\xi}) \right]_{\ell=1,\ldots,k} \rho(s; \hat{\xi})' Y_j(s) \lambda_0(s; \hat{\xi}) \, \mathrm{d}s \right\}; \\
\hat{\boldsymbol{\Delta}}_1(\mathrm{PW}_k) &= \frac{1}{n} \sum_{j=1}^{n} \exp\{\hat{\beta}' \boldsymbol{X}_j\} \left[\left(\frac{(R_{j\nu_j}^{\tau})^{\ell+1}}{\ell+1} \right)_{\ell=1,\ldots,k} \right] \boldsymbol{X}_j'; \\
\hat{\boldsymbol{\Sigma}}_{22} &= \frac{1}{2n} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{N_j(\tau)} \rho(W_{ji}; \hat{\xi})^{\otimes 2} + \exp\{\hat{\beta}' \boldsymbol{X}_j\} \int_0^{\tau} \rho(s; \hat{\xi})^{\otimes 2} Y_j(s) \lambda_0(s; \hat{\xi}) \, \mathrm{d}s \right\}; \\
\hat{\boldsymbol{\Sigma}}_{33} &= \frac{1}{2n} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{N_j(\tau)} \left[\boldsymbol{X}_j - \frac{\sum_{m=1}^{n} \boldsymbol{X}_m Y_m(W_{ji}) \exp\{\hat{\beta}' \boldsymbol{X}_m\}}{\sum_{m=1}^{n} Y_m(W_{ji}) \exp\{\hat{\beta}' \boldsymbol{X}_m\}} \right]^{\otimes 2} + \right. \\
&\qquad \left. \exp\{\hat{\beta}' \boldsymbol{X}_j\} \int_0^{\tau} \left[\boldsymbol{X}_j - \frac{\sum_{m=1}^{n} \boldsymbol{X}_m Y_m(s) \exp\{\hat{\beta}' \boldsymbol{X}_m\}}{\sum_{m=1}^{n} Y_m(s) \exp\{\hat{\beta}' \boldsymbol{X}_m\}} \right]^{\otimes 2} Y_j(s) \lambda_0(s; \hat{\xi}) \, \mathrm{d}s \right\}; \\
\hat{\boldsymbol{\Delta}}_2 &= \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{X}_j' \exp\{\hat{\beta}' \boldsymbol{X}_j\} \int_0^{\tau} \rho(s; \hat{\xi}) Y_j(s) \lambda_0(s; \hat{\xi}) \, \mathrm{d}s.
\end{aligned}$$

Recall in the above expressions that $\boldsymbol{\rho}(s; \xi) = \nabla_{\xi} \log \lambda_0(s; \xi)$. Note also that $\hat{\boldsymbol{\Sigma}}_{22}, \hat{\boldsymbol{\Sigma}}_{33}$, and $\hat{\boldsymbol{\Delta}}_2$ are unaffected or are invariant with respect to ψ. Thus, an asymptotic α-level "polynomial" test for H_0 for this generalized BBS model rejects H_0 whenever

$$S(\mathrm{PW}_k) \equiv \boldsymbol{Q}'(\mathrm{PW}_k) \, \hat{\boldsymbol{\Gamma}}^{-}(\mathrm{PW}_k) \, \boldsymbol{Q}(\mathrm{PW}_k) \geq \chi^2_{\hat{k}^*;\alpha},$$

where $\hat{k}^*$ is the rank of

$$\hat{\mathbf{\Gamma}}(\mathrm{PW}_k) = \hat{\mathbf{\Sigma}}_{11.2}(\mathrm{PW}_k) + \hat{\mathbf{\Upsilon}}(\mathrm{PW}_k)\hat{\mathbf{\Sigma}}_{33}^{-1}\hat{\mathbf{\Upsilon}}(\mathrm{PW}_k)'.$$

The Monte Carlo simulation results presented in Section 4 show that tests based on this polynomial specification are powerful omnibus tests. Another appealing property of polynomial based tests is the ease in which directional tests are obtained from the components of $S(\mathrm{PW}_k)$. For instance, consider the i^{th} component of $S(\mathrm{PW}_k)$ given by

$$S_i(\mathrm{PW}_k) = \frac{Q_i^2(\mathrm{PW}_k)}{\hat{\Gamma}_{ii}(\mathrm{PW}_k)} \tag{3.2}$$

where $\hat{\Gamma}_{ii}(\mathrm{PW}_k)$ is the $(i,i)^{\text{th}}$ element of $\hat{\mathbf{\Gamma}}(\mathrm{PW}_k)$. Note that for $i = 1, \ldots, k$, (3.2) is asymptotically χ_1^2-distributed under H_0. The statistic in (3.2) could be viewed as the i^{th} directional test statistic. The simulation results in Section 4 reveal that these directional tests are powerful for detecting specific departures from the null distribution.

It is also possible to obtain the expressions for the limiting variance $\mathbf{\Gamma}(\mathrm{PW}_k)$. As illustrated in Example 1, the limiting variance is a function of the probability of perfect repair $p(\cdot)$. However, since the probability of perfect repair is unknown, we used the estimated covariance matrix $\hat{\mathbf{\Gamma}}(\mathrm{PW}_k)$ in forming the test statistics for our simulation study.

4. Finite-Sample Properties

Whereas in the previous sections, we investigated the asymptotic properties of the proposed class of tests, in this section we examine its properties for small to moderate sample sizes. In order to investigate the achieved levels and powers of the tests, we performed a simulation study using the CPHM involving a BBS model with Bernoulli-distributed covariates. For the time-dependent probability of perfect repair, we used the function $p(t) = 1 - \exp(-\eta t),\ \eta > 0$. Note that this choice implies that the probability of perfect repair increases over time, an intuitive and feasible assumption. In addition, we assumed τ to be large enough so each component was observed until the time of its first perfect repair. Thus, the discussion of the simulation results that follow does not involve censoring. The issue of using the proposed tests in the presence of censored data will be dealt with in the next section. We point out, however, that the average number of failures per component was kept fairly constant across increasing and decreasing failure rate alternatives. The simulation programs were coded in FORTRAN and subroutines from the IMSL (1987) Math/Stat Library were used for random number generation and matrix inversion. The simulations were ran using DIGITAL Visual Fortran 6.0.

We present results for the omnibus tests based on a polynomial specification for $\boldsymbol{\psi}$ with $k = 1, 2, 3, 4$, and the directional tests $S_i(\mathrm{PW}_4)$ for $i = 1, 2, 3, 4$.

4.1 Achieved Significance Levels

In examining the achieved levels of the tests, we tested the null hypothesis that the initial distribution comes from an exponential distribution, i.e., $H_0 : \lambda(\cdot) \in \mathcal{C} \equiv \{\lambda_0(\cdot; \xi) = \xi : \xi \in (0, \infty)\}$. The simulation study involved 5000 replications of the following experiment. For each combination of sample size $n \in \{50, 100, 200\}$ and $\eta \in \{0.05, 0.10, 0.30\}$, we generated record values $\{W_{ji} : j = 1, \ldots, n; i \geq 1\}$ for a BBS model. The initial record values were generated from a unit exponential distribution. Subsequent record values were generated by utilizing the memoryless property of the exponential distribution. The covariates were generated from a Bernoulli distribution with success probability $p^* = 0.5$. The mode of repair to be performed at each failure time, W_{ji}, was determined by generating Uniform[0,1] variates u_{ji}. If $u_{ji} < p(W_{ji})$, a perfect repair is assumed to have been performed at W_{ji}; otherwise, a minimal repair is assumed. Furthermore, we assumed τ to be large enough so each unit was observed until the time of its first perfect repair which we denoted by $W_{j\nu_j}$. The sample realization $\{w_{ji} : j = 1, \ldots, n; i = 1, \ldots, \nu_j\}$ was then utilized to test H_0 using the proposed tests. Simulations were performed at the 5% and 10% asymptotic levels, but since the results led to similar conclusions we discuss only results for the 5% asymptotic level tests.

Examining Table 1 for the various combinations of sample size and probability of perfect repair, the achieved levels of all the tests are consistent with the 5% level. We also performed simulations using the uniformity-based test and the spacings-based test introduced in Peña (1998a) which resulted in generalizations of the Horowitz-Neumann (1992) statistic and the Barlow, et al. (1972) statistic, respectively. To facilitate comparison, we adopt the same notation as in Peña (1998a). Thus, we use $S(\mathrm{GHN2})$ and $S(\mathrm{GB4})$ for the uniformity-based and spacings-based test statistics, respectively. The achieved levels of the uniformity-based test were comparable to those based on the polynomial choice. The spacings-based test, however, turned out to be anticonservative for small sample sizes. The anticonservatism decreased as sample size increased and as the parameter η in the probability of perfect repair, $p(t) = 1 - \exp(-\eta t)$, decreased. The latter result should be expected since a smaller η translated to more observations. For instance, for $\eta = 0.30$, the mean number of repairs per component is roughly 3, while for $\eta = 0.05$, the corresponding mean turned out to be approximately 6.

We point out that using the expression given by (2.4) as the estimator of $\hat{\boldsymbol{\Sigma}}$ significantly improved the results. Previous simulations where the covariance matrix was estimated using only the predictable covariation process (Peña, 1998a; Agustin and Peña, 2000) resulted in anticonservative tests. The use of the estimator based on a combination of the optional and predictable covariation processes shows promise and calls for further investigation.

We turn our attention to the achieved powers of the proposed tests. For the power simulations in the next section, we used $n = 200$ and $\eta = 0.10$ since for this combination the achieved levels of all the tests are consistent with the nominal asymptotic level of 5%.

4.2 Achieved Powers

To investigate the powers of the different tests, we retained the exponential null hypothesis and considered Weibull and gamma alternatives for the initial distribution of failure ages. Hence, given values of the shape parameter γ and the scale parameter β, record values for each alternative were generated via $W_{ji} = \Lambda^{-1}(X_{j1} + X_{j2} + \ldots + X_{ji})$, where $X_{j\ell}$'s are unit exponential random variables and Λ is the hazard function corresponding to the alternative. The simulations were done for both the 5% and 10% asymptotic level tests. As in the discussion for the achieved significance levels we simply focus on the 5% level tests.

For the Weibull-type alternatives, the simulated powers as the shape parameter varies are plotted in Figure 1. Recall that $\gamma < 1$ yields increasing failure rate alternatives, while $\gamma > 1$ results in decreasing failure rate alternatives. Notice that for these alternatives, all the omnibus tests performed remarkably well. A closer examination of Figure 1 reveals that the test based on $S(\mathrm{PW}_1)$ had the highest power for increasing failure rate alternatives, while the test based on $S(\mathrm{PW}_2)$ came out as the best for decreasing failure rate alternatives. The directional tests based on $S_i(\mathrm{PW}_4), i = 1, 2, 3, 4$, behave differently. Note that the achieved powers are decreasing as i increases from 1 to 4. Moreover, the achieved powers are higher for increasing failure rate alternatives indicating that these directional tests are more sensitive to increasing failure rate Weibull alternatives. We also performed simulations for values of $\beta \neq 1$ and found the results to be consistent with those presented in Figure 1. This result should not come as a surprise since our composite null hypothesis does not postulate any value for β.

For the gamma-type alternatives, the simulated power results are plotted in Figure 2. The omnibus test based on $S(\mathrm{PW}_4)$ emerged as the best for the increasing failure rate alternatives while the test based on $S(\mathrm{PW}_3)$ came out to be the most powerful for decreasing failure rate alternatives. Examining the performance of the directional tests, we again notice that the achieved powers are decreasing as i increases from 1 to 4, with the achieved power of the test based on $S_1(\mathrm{PW}_4)$ substantially higher than the others. In addition, the achieved powers are higher for the decreasing failure rate alternatives.

For both types of alternatives, we also performed power simulations for the uniformity-based and spacings-based tests statistics. The achieved powers of these two tests are slightly higher than those of the tests based on the polynomial specification. The main advantage of the polynomial-based tests, however, is their simplicity in terms of implementation.

The results of the simulation study suggest that the choice of the smoothing order k is critical and depends on the alternative that one wishes to detect. In the event that one does not have an idea on the type of alternative that might arise when H_0 does not hold, then for the generalized recurrent event setting with covariates, tests based on the polynomial specification with $k \in \{3, 4\}$ are the recommended omnibus tests. An important and natural direction for future work is the development of a methodology for choosing adaptively or in a data-dependent manner the smoothing order k. Research on this aspect for this hazard-based formulation is currently on-going; while for the classical density-based formulation, papers of Ledwina (1994) and Kallenberg and Ledwina (1995) have addressed this issue using Schwarz information criterion.

5. Illustrative Applications to Real Data

In medical studies, censored single spell data are widely encountered. Single spell data can be modeled using a special case of the BBS model by taking the probability of perfect repair to be identically one. In this section, we illustrate the applicability of our proposed tests to right-censored single spell data. To this end, the processes that are of interest to us are $\{(N_j(\cdot), Y_j(\cdot)) : j = 1, \ldots, n\}$ where

$$N_j(t) = I\{Z_j \leq t, \delta_j = 1\} \quad \text{and} \quad Y_j(t) = I\{Z_j \geq t\},$$

with Z_j being the minimum of the failure time and the censoring variable for the j^{th} unit, and δ_j is the censoring indicator. Keeping in mind the definition of these processes, it is a straightforward exercise to obtain the score statistic and estimated covariance matrix corresponding to the polynomial specification. In fact, the expressions are analogous to the ones presented in Section 3.

For a concrete example, consider the Stanford Heart Transplant data set given in Table 7.1

of Andrews and Herzberg (1985). This data set consists of 184 cases which represent those patients who received a transplant. Of the 184 cases, 71 are right-censored. The variable of interest is the survival time, in days, after transplant, while the covariate is the age, in years, at transplant. Assuming the CPHM, we utilized the partial likelihood to obtain $\hat{\beta} = 0.029$ which turned out to be significant at the 0.01 level. Note that our main interest in this paper is goodness-of-fit testing for the baseline hazard rate function. Thus, we proceeded to test the hypothesis that the baseline hazard rate function is constant, that is, $H_0 : \lambda(\cdot) \in \mathcal{C} \equiv \{\lambda_0(\cdot;\xi) = \xi : \xi \in (0,\infty)\}$. Using $\hat{\beta}$ in the profile likelihood yielded $\hat{\xi} = 0.000263$. Plugging-in these estimates in place of the unknown parameters and taking into account the necessary adjustments, the resulting test statistics and their corresponding p-values are as follows: $S(\mathrm{PW}_1) = S_1(\mathrm{PW}_4) = 13.52 (p = .0002)$, $S(\mathrm{PW}_2) = 16.12(p = .0003)$, $S(\mathrm{PW}_3) = 17.17(p = .0007)$, $S(\mathrm{PW}_4) = 17.33(p = .0017)$, $S_2(\mathrm{PW}_4) = 8.31(p = .0039)$, $S_3(\mathrm{PW}_4) = 5.12(p = .0237)$, and $S_4(\mathrm{PW}_4) = 3.50(p = .0614)$. Examining these values, notice that all the omnibus tests rejected the null hypothesis of constant baseline hazard rate function at the 1% level. Of the directional tests, the tests based on $S_1(\mathrm{PW}_4)$ (equivalent to $S(\mathrm{PW}_1)$) and $S_2(\mathrm{PW}_4)$ rejected H_0 at the 1% level, while the one based on $S_3(\mathrm{PW}_4)$ rejected H_0 at the 5% level.

To illustrate the applicability of the proposed procedures to recurrent event models, we analyzed the data set presented as Table 2.7 in Blischke and Prabhakar Murthy (2000). The data set consists of time between successive failures over a two-year period for the hydraulic systems of six large load-haul-dump (LHD) machines. LHD machines are used for moving ore and rock. In terms of equipment reliability, the hydraulic system is a vital subsystem of LHD machines. For the analysis, we used the BBS model for the successive failure times. We treated all the intermediate repairs as minimal repairs and the final repair as a perfect repair. In addition to the failure times, information regarding the age of the machines are also available. However, in contrast to the Stanford Heart Transplant data set where the actual ages are known, this data set only classifies the machines as old, medium age, and relatively new. For our analysis, we treated age as a covariate. Due to the nature of the classification, we used two binary covariates, X_1 and X_2. Thus the old machines are associated with $X_1 = 1$ and $X_2 = 0$; the medium age ones with $X_1 = 0$ and $X_2 = 1$; and the relatively new ones with $X_1 = 0$ and $X_2 = 0$. Assuming the CPHM and using only the partial likelihood, we obtained the estimates $\hat{\beta}_1 = 0.0541$ and $\hat{\beta}_2 = -0.1028$. The hypotheses of interest are $H_0 : \lambda(\cdot) \in \mathcal{C} \equiv \{\lambda_0(\cdot;\xi) = \xi : \xi \in (0,\infty)\}$ versus $H_0 : \lambda(\cdot) \notin \mathcal{C}$. To estimate ξ, we used the estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ in the profile likelihood and obtained $\hat{\xi} = 0.0077$. The observed values of the test statistics and their associated p-values are: $S(\mathrm{PW}_1) = S_1(\mathrm{PW}_4) = 7.4503 (p = .0063)$, $S(\mathrm{PW}_2) = 9.9551 (p = .0069)$, $S(\mathrm{PW}_3) = 9.8112 (p = .0202)$, $S(\mathrm{PW}_4) = 9.9291 (p = .0416)$, $S_2(\mathrm{PW}_4) = 3.8127 (p = .0509)$, $S_3(\mathrm{PW}_4) = 2.4164 (p = .1201)$, and $S_4(\mathrm{PW}_4) = 1.8909 (p = .1691)$. All the omnibus tests rejected the null hypothesis of a constant baseline hazard rate function, with $S(\mathrm{PW}_1)$ and $S(\mathrm{PW}_2)$ rejecting H_0 at the 1% level, and the latter two rejecting H_0 at the 5% level. For the directional tests, however, only $S_1(\mathrm{PW}_4)$ and $S_2(\mathrm{PW}_4)$ rejected H_0 at the 5% level.

These illustrative examples demonstrate the wide applicability of the proposed tests. In particular, we were able to show that our family of tests are applicable to both single spell and recurrent event data as well as to different covariate structures. The hazard-based formulation in conjunction with the counting process and martingale framework adopted in this paper enabled us to develop goodness-of-fit procedures that are adaptable to various situations including, and more importantly, recurrent event models.

6. Appendix: Regularity Conditions

We enumerate here the regularity conditions which are needed for the asymptotic properties of the estimated score function to hold. These conditions are:

(I) There exists a neighborhood $\mathcal{C}_0$ of ξ_0 such that on $\mathcal{T} \times \mathcal{C}_0$, $\lambda_0(t;\xi) > 0$, and the partial derivatives $\frac{\partial}{\partial \xi_k}\lambda_0(s;\xi)$, $\frac{\partial^2}{\partial \xi_k \partial \xi_\ell}\lambda_0(s;\xi)$, and $\frac{\partial^3}{\partial \xi_k \partial \xi_\ell \mathrm{d}\xi_m}\lambda_0(s;\xi)$, $k, \ell, m = 1, \ldots, p$, exist and are continuous at $\xi = \xi_0$.

(II) $\int_0^\tau \lambda_0(s;\xi)\ \mathrm{d}s < \infty$ for $\xi \in \mathcal{C}_0$.

(III) There exists a neighborhood $\mathcal{B}$ of β_0 such that on $\mathcal{T} \times \mathcal{C}_0 \times \mathcal{B}$, the log-likelihood process

$$\begin{aligned}\ell^F(t;\boldsymbol{\theta},\xi,\beta) &= \sum_{j=1}^n \int_0^t \log[Y_j(s)\lambda_0(s;\xi)\exp\{\beta'\boldsymbol{X}_j(s)\}\exp\{\boldsymbol{\theta}'\boldsymbol{\psi}(s;\xi)\}]\ \mathrm{d}N_j(s) - \\ &\quad \sum_{j=1}^n \int_0^t Y_j(s)\lambda_0(s;\xi)\exp\{\beta'\boldsymbol{X}_j(s)\}\exp\{\boldsymbol{\theta}'\boldsymbol{\psi}(s;\xi)\}\ \mathrm{d}s\end{aligned}$$

may be differentiated three times with respect to ξ and twice with respect to β, and the order of integration and differentiation can be interchanged.

(IV) There exist functions $\boldsymbol{s}_{(m)},\ (m=0,1,2)$, and $\boldsymbol{p}_{(m)},\ (m=1,2)$ with domain $\mathcal{T} \times \mathcal{C}_0 \times \mathcal{B}$ such that, as $n \to \infty$,

$$\begin{aligned}\sup_{t\in\mathcal{T};\beta\in\mathcal{B}} \|\boldsymbol{S}_{(m)}(t;\beta) - \boldsymbol{s}_{(m)}(t;\beta)\| &\xrightarrow{pr} 0, \quad (m=0,1,2); \\ \sup_{t\in\mathcal{T};\xi\in\mathcal{C}_0;\beta\in\mathcal{B}} \|\boldsymbol{P}_{(m)}(t;\xi,\beta) - \boldsymbol{p}_{(m)}(t;\xi,\beta)\| &\xrightarrow{pr} 0, \quad (m=1,2).\end{aligned}$$

(V) The functions $\boldsymbol{s}_{(m)}, (m=0,1,2)$, are bounded on $\mathcal{T} \times \mathcal{B}$; the family $\{\boldsymbol{s}_{(m)}(t;\cdot) : t \in \mathcal{T}\}$ is equicontinuous at β_0; and $s_{(0)}$ is bounded away from zero on $\mathcal{T} \times \mathcal{B}$. Moreover, the functions $\boldsymbol{p}_{(m)}, (m=1,2)$ are bounded on $\mathcal{T} \times \mathcal{C}_0 \times \mathcal{B}$; and the family $\{\boldsymbol{p}_{(m)}(t;\cdot,\cdot) : t \in \mathcal{T}\}$ is equicontinuous at (ξ_0,β_0).

(VI) For all $t \in \mathcal{T}$ and $\beta \in \mathcal{B}$, $\frac{\partial}{\partial\beta} s_{(0)}(t;\beta) = \boldsymbol{s}_{(1)}(t;\beta)$ and $\frac{\partial^2}{\partial\beta\partial\beta'} s_{(0)}(t;\beta) = \boldsymbol{s}_{(2)}(t;\beta)$.

(VII) The matrix

$$\boldsymbol{\Sigma}(\xi_0,\beta_0) = \begin{bmatrix} \boldsymbol{\Sigma}_{11}(\xi_0,\beta_0) & \boldsymbol{\Sigma}_{12}(\xi_0,\beta_0) & \mathbf{0} \\ \boldsymbol{\Sigma}_{21}(\xi_0,\beta_0) & \boldsymbol{\Sigma}_{22}(\xi_0,\beta_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_{33}(\xi_0,\beta_0) \end{bmatrix} \tag{6.1}$$

is positive definite, where the component matrices are defined in subsection 2.2.

(VIII) There exists a $\delta > 0$ such that

$$\frac{1}{\sqrt{n}} \sup_{t\in\mathcal{T};\ j=1,\ldots,n} |\boldsymbol{X}_j(t)| Y_j(t) I\{\beta_0{}'\boldsymbol{X}_j(t) > -\delta|\boldsymbol{X}_j(t)|\} \xrightarrow{pr} 0,$$

and for each $\epsilon > 0$ and $t \in \mathcal{T}$,

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^t |\boldsymbol{\psi}(s;\xi_0)|^2 I\{|\boldsymbol{\psi}(s;\xi_0)| \geq \sqrt{n}\epsilon\} Y_j(s)\exp\{\beta_0{}'\boldsymbol{X}_j(s)\}\lambda_0(s;\xi_0)\mathrm{d}s \xrightarrow{pr} 0;$$

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^t |\rho(s;\xi_0)|^2\, I\left\{|\rho(s;\xi_0)| > \sqrt{n}\epsilon\right\} Y_j(s)\exp\{\beta_0{}'\boldsymbol{X}_j(s)\}\lambda_0(s;\xi_0)\mathrm{d}s \xrightarrow{pr} 0.$$

(IX) There exist functions J_i, $(i = 1, 2, 3)$, and K defined on $\mathcal{T}$ such that for $k, \ell, m = 1, \ldots, p$, and $t \in \mathcal{T}$,

$$\sup_{\xi\in\mathcal{C}_0}\left|\frac{\partial}{\partial\xi_k}\lambda_0(t;\xi)\right| \leq J_1(t); \qquad \sup_{\xi\in\mathcal{C}_0}\left|\frac{\partial^2}{\partial\xi_k\partial\xi_\ell}\lambda_0(t;\xi)\right| \leq J_2(t);$$

$$\sup_{\xi\in\mathcal{C}_0}\left|\frac{\partial^3}{\partial\xi_k\partial\xi_\ell\partial\xi_m}\lambda_0(t;\xi)\right| \leq J_3(t); \qquad \sup_{\xi\in\mathcal{C}_0}\left|\frac{\partial^3}{\partial\xi_k\partial\xi_\ell\partial\xi_m}\log\lambda_0(t;\xi)\right| \leq K(t).$$

Moreover,

$$\int_0^\tau J_i(s)\boldsymbol{S}_{(3-i)}(s;\beta_0)\ \mathrm{d}s,\ i = 1, 2, 3,$$

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau K(s)Y_j(s)\exp\{\beta_0{}'\boldsymbol{X}_j(s)\}\lambda_0(s;\xi_0)\ \mathrm{d}s,\ \text{and}$$

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau \left|\frac{\partial^2}{\partial\xi_k\partial\xi_\ell}\log\lambda_0(s;\xi_0)\right|^2 Y_j(s)\exp\{\beta_0{}'\boldsymbol{X}_j(s)\}\lambda_0(s;\xi_0)\ \mathrm{d}s$$

all converge in probability to finite quantities.

(X) On $\mathcal{T} \times \mathcal{C}_0 \times \mathcal{B}$, the partial derivatives $\frac{\partial}{\partial\xi}\boldsymbol{\psi}(t;\xi)$ and $\frac{\partial^2}{\partial\xi\partial\xi'}\boldsymbol{\psi}(t;\xi)$ exist and are continuous at (ξ_0, β_0). Furthermore, the processes $\{\boldsymbol{\psi}(t;\xi_0) : t \in \mathcal{T}\}$ and $\{\frac{\partial}{\partial\xi}\boldsymbol{\psi}(t;\xi) : t \in \mathcal{T}\}$ are locally bounded and predictable.

(XI) Let $\boldsymbol{R}^\xi(t;\xi,\beta) = \frac{1}{n}\sum_{j=1}^{n}\int_0^t \frac{\partial}{\partial\xi}\boldsymbol{\psi}(s;\xi,\beta)Y_j(s)\exp\{\beta'\boldsymbol{X}_j(s)\}\ \mathrm{d}s$. There exists a function $\boldsymbol{r}^\xi$ defined on $\mathcal{T} \times \mathcal{C}_0 \times \mathcal{B}$ such that

$$\sup_{t\in\mathcal{T};\xi\in\mathcal{C}_0;\ \beta\in\mathcal{B}} \|\boldsymbol{R}^\xi(t;\xi,\beta) - \boldsymbol{r}^\xi(t;\xi,\beta)\| \xrightarrow{pr} 0.$$

Furthermore, for $i = 1, \ldots, p$, and $\ell = 1, \ldots, k$,

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau \left|\frac{\partial}{\partial\xi_i}\psi_\ell(s;\xi_0)\right|^2 Y_j(s)\exp\{\beta_0{}'\boldsymbol{X}_j(s)\}\lambda_0(s;\xi_0)\ \mathrm{d}s = O_p(1).$$

(XII) The family $\{\boldsymbol{r}^\xi(t;\cdot,\cdot) : t \in \mathcal{T}\}$ is equicontinuous at (ξ_0, β_0).

References

Aalen, O. (1978). Statistical inference for a family of counting processes. *Annals of Statistics*, **6**, 701–726.

Agustin, Z. and Peña, E. (2001) Goodness-of-fit of the distribution of time-to-first-occurrence in recurrent event models. *Lifetime Data Analysis*, **7**, 289–306.

Andersen, P. and Gill, R. (1982) Cox's regression model for counting processes: A large sample study. *The Annals of Statistics*, **10**, 1100–1120.

Andersen, P., Borgan, O., Gill, R. and Keiding, N. (1993). *Statistical Models Based on Counting Processes.* New York: Springer-Verlag.

Andrews, D. and Herzberg, A. (1985) *Data: A collection of problems from many fields for the student and research worker.* New York: Springer-Verlag.

Baltazar-Aban, I. and Peña, E. (1995) Properties of hazard-based residuals and implications in model diagnostics. *Journal of the American Statistical Association*, **90**, 185–197.

Barlow, R., Bartholomew, D., Bremmer, J. and Brunk, H. (1972) *Statistical Inference Under Order Restrictions.* New York: Wiley.

Bickel, P. and Ritov, Y. (1992) Testing for goodness of fit: a new approach. In *Nonparametric Statistics and Related Topics* (ed. A.K. Md. E. Saleh), pp. 51–57. Elsevier Science Publishers B.V.

Blischke, W. and Prabhakar Murthy, D. (2000) *Reliability Modeling, Prediction, and Optimization.* New York: Wiley.

Block, H., Borges, W. and Savits, T. (1985) Age-dependent minimal repair. *Journal of Applied Probability*, **22**, 370–385.

Borgan, O. (1984) Maximum likelihood estimation in parametric counting process models, with applications to censored failure time data. *Scandinavian Journal of Statistics*, **11**, 1–16.

Brown, M. and Proschan, F. (1983) Imperfect repair. *Journal of Applied Probability*, **20**, 851–859.

Choi, S., Hall, W. and Schick, A. (1996) Asymptotically uniformly most powerful tests in parametric and semiparametric models. *The Annals of Statistics*, **24**, 841–861.

Cox, D. (1972) Regression models and life-tables (with discussion). *Journal of the Royal Statistical Society B*, **34**, 187–220.

Cox, D. and Hinkley, D. (1974) *Theoretical Statistics.* London: Chapman and Hall.

Eubank, R. and Hart, J. (1992) Testing goodness-of-fit in regression via order selection criteria. *The Annals of Statistics*, **20**, 1412–1425.

Eubank, R. and LaRiccia, V. (1992) Asymptotic comparison of Cramer-von Mises and nonparametric function estimation techniques for testing goodness-of-fit. *The Annals of Statistics*, **20**, 2071–2086.

Fan, J. (1996) Test of significance based on wavelet thresholding and Neyman's truncation. *Journal of the American Statistical Association*, **91**, 674–688.

Fleming, T. and Harrington, D. (1991) *Counting Processes and Survival Analysis.* New York: Wiley.

Gray, R. and Pierce, D. (1985) Goodness-of-fit for censored survival data. *The Annals of Statistics*, **13**, 552–563.

Horowitz, J. and Neumann, G. (1992) A generalized moments specification test of the proportional hazards model. *Journal of the American Statistical Association*, **87**, 234–240.

IMSL, Inc. (1987) *Math/Stat Library*. Houston: Authors.

Inglot, T., Kallenberg, W. and Ledwina, T. (1994) Power approximations to and power comparison of smooth goodness-of-fit tests. *Scandinavian Journal of Statistics*, **21**, 131–145.

Kallenberg, W. and Ledwina, T. (1995) Consistency and Monte Carlo simulation of a data driven version of smooth goodness-of-fit tests. *The Annals of Statistics*, **23**, 1594–1608.

Keiding, N., Andersen, C. and Fledelius, P. (1998) The Cox regression model for claims data in non-life insurance. *Astin Bulletin*, **28**, 95–118.

Kopecky, K. and Pierce, D. (1979) Efficiency of smooth goodness-of-fit tests. *Journal of the American Statistical Association*, **74**, 393–397.

Lagakos, S. (1981) The graphical evaluation of explanatory variables in proportional hazard regresion models. *Biometrika*, **68**, 93–98.

Ledwina, T. (1994) Data driven version of Neyman's smooth test of fit. *Journal of the American Statistical Association*, **89**, 1000–1005.

Li, G. and Doss, H. (1993) Generalized Pearson-Fisher chi-square goodness-of-fit tests, with applications to models with life history data. *The Annals of Statistics*, **21**, 772–797.

Lin, D. and Spiekerman, C. (1996) Model-checking techniques for parametric regression with censored data. *Scandinavian Journal of Statistics*, **23**, 157–177.

Miller, F. and Quesenberry, G. (1979) Power studies of tests for uniformity II. *Communications in Statistics B - Simulation and Computation*, **8**, 271–290.

Neyman, J. (1937) "Smooth test" for goodness of fit. *Skand. Aktuarietidskrift*, **20**, 149–199.

Pearson, K. (1900) On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can reasonably be supposed to have arisen from random sampling. *Philos. Mag. Ser.*, **5**, 157–175.

Peña, E. (1995) Residuals from Type II censored samples. In *Recent Advances in Life-Testing and Reliability*, ed. N. Balakrishnan, Boca Raton, FL: CRC Press, pp. 523–543.

Peña, E. (1998a) Smooth goodness-of-fit tests for the baseline hazard in Cox's proportional hazards model. *Journal of the American Statistical Association*, **93**, 673–692.

Peña, E. (1998b) Smooth goodness-of-fit tests for composite hypothesis in hazard-based models. *The Annals of Statistics*, **26**, 1935–1971.

Peña, E. and Agustin, Z. A general asymptotic result useful in models with nuisance parameters. Technical Report No. 204 (May 2001), Department of Statistics, University of South Carolina, Columbia, South Carolina.

Pierce, D. (1982) The asymptotic effect of substituting estimators for parameters in certain types of statistics. *The Annals of Statistics*, **10**, 475–478.

Randles, R. (1982) On the asymptotic normality of statistics with estimated parameters. *The Annals of Statistics*, **10**, 462–474.

Randles, R. (1984) On tests applied to residuals. *Journal of the American Statistical Association*, **79**, 349–354.

Rayner, J. and Best, D. (1989) *Smooth Tests of Goodness of Fit.* New York: Oxford.

Ying, Z., Wei, L. and Lin, J. (1992) Prediction of survival probability based on a linear regression model. *Biometrika*, **79**, 205–209.

Table 1: Simulated levels of 5%-asymptotic level tests for different sample sizes, n, and probability of perfect repair, $p(t) = 1 - e^{-\eta t}$. The failure times under the null hypothesis were generated according to a BBS(1985) model with initial distribution EXP(1). The covariate was assumed to be Bernoulli-distributed with success probability $p^* = 0.50$.

η	0.30			0.10			0.05		
n	50	100	200	50	100	200	50	100	200
Test									
$S(\text{PW}_1)$	4.64	5.00	4.30	4.56	5.22	4.86	4.72	4.74	5.02
$S(\text{PW}_2)$	3.96	3.70	3.70	4.38	4.30	4.30	4.62	4.28	4.88
$S(\text{PW}_3)$	4.82	4.38	4.28	4.26	4.26	4.36	4.32	4.36	4.86
$S(\text{PW}_4)$	5.06	5.48	5.08	5.04	5.16	4.50	5.22	4.78	4.78
$S_1(\text{PW}_4)$	4.64	5.00	4.30	4.56	5.22	4.86	4.72	4.74	5.02
$S_2(\text{PW}_4)$	4.24	4.26	4.14	4.08	4.74	4.50	4.28	4.52	4.56
$S_3(\text{PW}_4)$	4.04	3.64	3.34	3.52	4.10	4.04	3.82	4.08	4.10
$S_4(\text{PW}_4)$	4.00	3.42	3.30	3.36	3.54	3.56	3.62	3.66	3.42
$S(\text{GHN2})$	4.60	4.70	5.10	4.82	5.10	4.74	5.20	4.66	4.94
$S(\text{GB4})$	6.36	5.34	4.88	5.76	5.92	4.88	5.50	4.90	4.90

Table 2: Simulated powers of the 5%-asymptotic level tests when the failure times were generated according to a BBS (1985) model with initial distribution Weibull(γ, β), probability of perfect repair $p(t) = 1 - e^{-0.10t}$, and number of components $n = 200$. The covariate was assumed to be Bernoulli-distributed with success probability $p^* = 0.50$.

Test	β	1	1	1	1	1	1	1
	γ	0.85	0.90	0.95	1.00	1.05	1.10	1.15
$S(\mathrm{PW}_1)$		99.74	87.32	30.62	5.32	23.22	65.86	93.04
$S(\mathrm{PW}_2)$		99.76	85.58	24.88	4.90	24.54	68.92	94.60
$S(\mathrm{PW}_3)$		99.82	84.38	24.72	4.80	18.38	62.60	92.78
$S(\mathrm{PW}_4)$		99.66	79.64	20.28	5.50	21.08	66.04	93.68
$S_1(\mathrm{PW}_4)$		99.74	87.32	30.62	5.32	23.22	65.86	93.04
$S_2(\mathrm{PW}_4)$		96.32	67.78	21.26	5.36	13.40	39.78	71.12
$S_3(\mathrm{PW}_4)$		85.42	49.50	15.80	4.66	6.70	18.30	38.14
$S_4(\mathrm{PW}_4)$		68.26	37.08	12.38	4.14	2.68	6.40	13.46
$S(\mathrm{GHN2})$		99.94	92.58	37.50	5.48	34.50	82.74	98.50
$S(\mathrm{GB4})$		99.94	92.64	38.26	5.56	28.72	76.02	97.10

Table 3: Simulated powers of the 5%-asymptotic level tests when the failure times were generated according to a BBS model with initial distribution Gamma(γ, β), probability of perfect repair $p(t) = 1 - e^{-0.10t}$ and number of components $n = 200$. The covariate was assumed to be Bernoulli-distributed with success probability $p^* = 0.50$.

Test	β	1	1	1	1	1	1	1	1	1	1	1
	γ	0.70	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.30	1.50
$S(\mathrm{PW}_1)$		86.26	28.24	14.20	6.72	4.92	7.42	16.04	28.44	43.22	71.80	97.60
$S(\mathrm{PW}_2)$		94.96	36.10	18.84	7.78	4.32	6.14	13.98	27.20	43.78	77.48	99.38
$S(\mathrm{PW}_3)$		95.12	31.64	14.48	6.28	4.42	7.22	16.26	31.38	50.18	84.20	99.84
$S(\mathrm{PW}_4)$		97.58	38.46	18.26	7.60	4.66	5.98	13.08	26.34	44.76	80.72	99.76
$S_1(\mathrm{PW}_4)$		86.26	28.24	14.20	6.72	4.92	7.42	16.04	28.44	43.22	71.80	97.60
$S_2(\mathrm{PW}_4)$		48.32	12.34	7.28	4.62	5.04	6.38	10.66	17.46	24.88	44.42	77.30
$S_3(\mathrm{PW}_4)$		20.16	5.64	3.72	3.60	4.78	5.68	8.24	12.38	16.16	28.06	53.82
$S_4(\mathrm{PW}_4)$		7.52	2.36	2.28	2.60	4.16	4.98	6.82	9.98	12.16	19.80	37.00
$S(\mathrm{GHN2})$		99.64	60.04	32.58	11.10	5.00	9.12	25.58	48.08	69.28	94.68	99.98
$S(\mathrm{GB4})$		96.12	39.40	19.46	7.86	5.66	8.92	21.66	39.12	57.30	86.46	99.74

Figure 1: Plots of the simulated powers as the shape parameter of the alternative Weibull distribution varies. **Legend:** Solid = SPW1 and S1PW; Dots = SPW2; DotDash = SPW3; ShortDash = SPW4; DotDotDash = S2PW; AltDash = S3PW; MedDash = S4PW. The horizontal solid line represents the value of 5%, the desired level.

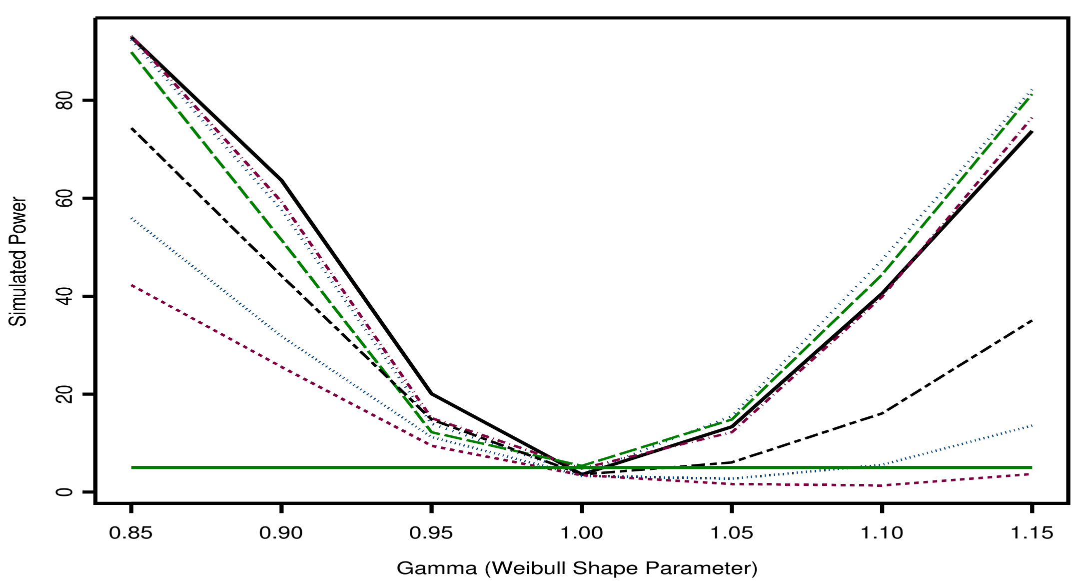

Figure 2: Plots of the simulated powers as the shape parameter of the alternative gamma distribution varies. **Legend:** Solid = SPW1 and S1PW; Dots = SPW2; DotDash = SPW3; ShortDash = SPW4; DotDotDash = S2PW; AltDash = S3PW; MedDash = S4PW. The horizontal solid line represents the value of 5%, the desired level.