Goodness-of-Fit Tests With Right-Censored Data

by

Edsel A. Peña
Department of Statistics
University of South Carolina

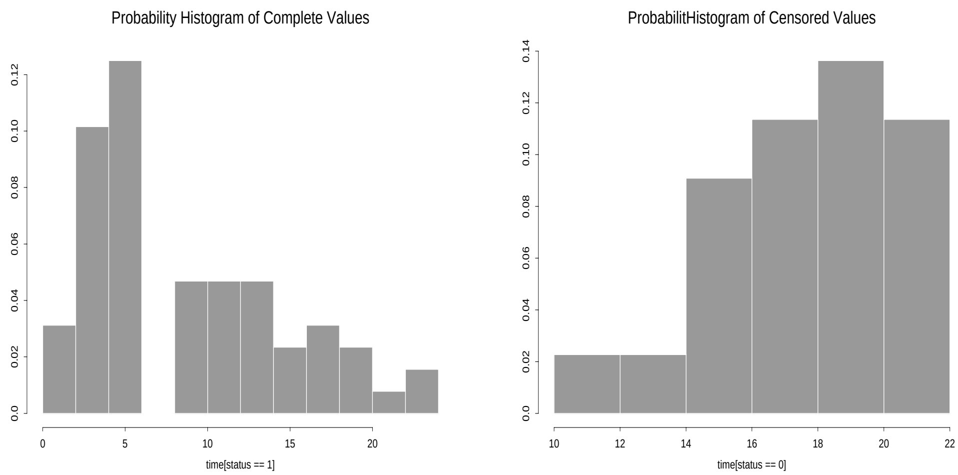
Colloquium Talk
August 31, 2000

Research supported by an NIH Grant

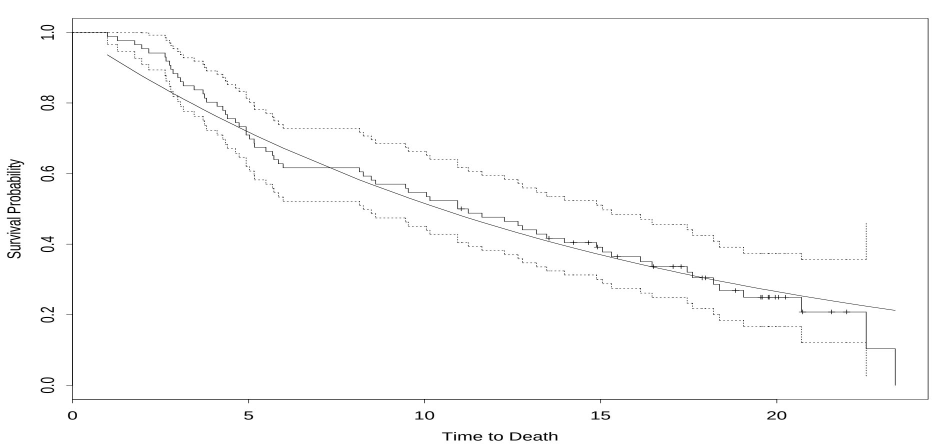
1. Practical Problem

- Right-censored survival data for lung cancer patients from Gatsonis, Hsieh and Korwar (1985) with 86 observations (63 complete and 23 right-censored).

- Probability histograms of the Complete and Censored Values



- Product-limit estimator and best-fitting exponential.



- **Question:** Did the survival data come from the family of exponential distributions? Or was it from the family of Weibull distributions?

2. On Densities and Hazards

- T = a positive-valued continuous failure-time variable, e.g.,
  - time-to-failure of a mechanical or electronic system
  - time-to-occurrence of an event
  - survival time of a patient in a clinical trial
- $f(t)$ = density function of T. Practical interpretation:

$$f(t)\Delta t \approx \mathbf{P}\left\{T \in [t, t+\Delta t)\right\}.$$

- $F(t) = \mathbf{P}\{T \leq t\}$ = distribution function
- $\bar{F}(t) = 1 - F(t)$ = survivor function
- $\lambda(t) = \frac{f(t)}{\bar{F}(t)}$ = hazard rate function. Practical interpretation:

$$\lambda(t)\Delta t \approx \mathbf{P}\left\{T \in [t, t+\Delta t) | T \geq t\right\}.$$

- $\Lambda(t) = \int_0^t \lambda(w)\mathrm{d}w = -\log[\bar{F}(t)]$ = (cumulative) hazard function
- Equivalences:

$$\bar{F}(t) = e^{-\Lambda(t)}$$

$$f(t) = \lambda(t)e^{-\Lambda(t)}$$

- Two Simple Examples:

  ◇ Exponential:

$$f(t;\eta) = \eta e^{-\eta t}$$

$$\bar{F}(t;\eta) = e^{-\eta t}$$

$$\lambda(t;\eta) = \eta$$

$$\Lambda(t;\eta) = \eta t$$

  ◇ Two-Parameter Weibull:

$$f(t;\alpha,\eta) = (\alpha\eta)(\eta t)^{\alpha-1} e^{-(\eta t)^{\alpha}}$$

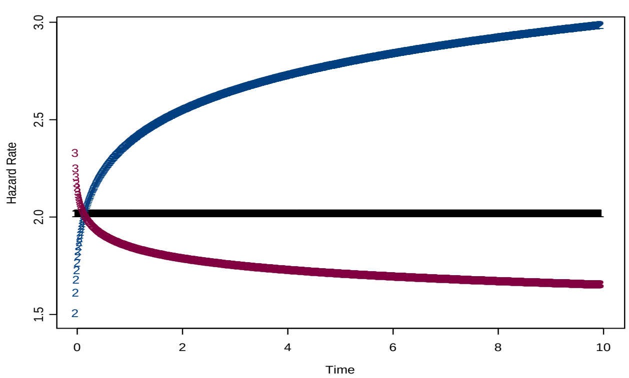
$$\bar{F}(t;\alpha,\eta) = e^{-(\eta t)^{\alpha}}$$

$$\lambda(t;\alpha,\eta) = (\alpha\eta)(\eta t)^{\alpha-1}$$

$$\Lambda(t;\alpha,\eta) = (\eta t)^{\alpha}$$

  ◇ Qualitative Aspects from Plots of Hazards

Figure 1: Weibull Hazard Plots



3. On Hazard-Based Modeling

- Advantages of specifying models via hazards:
  - ◇ Vantage point in density modeling: Time origin. ['What proportion are going to fail in $[t, t+\Delta t)$?'].
  - ◇ Vantage point in hazard modeling: 'Present, together with information that accumulated in the past.' ['Given history until time t, what proportion are going to fail among those at risk in $[t, t+\Delta t)$?'].
  - ◇ Qualitative aspects (e.g., IFR, or bath-tub) can be incorporated.
  - ◇ Incorporates dynamic evolution. Relevant in reliability systems modeling where failure rates of components of a system may drastically change due to the failure of other components (Arjas and Norros; Lawless; Hollander and Peña, Lynch and Padgett, etc.).
  - ◇ Likelihood construction natural via product integrals.
  - ◇ Adapts well in the presence of right-censored or truncated data.
  - ◇ Conducive to modeling with point processes (popularized by Aalen; Andersen and Gill; etc.).

- Theory to be presented applicable to more general models, but will only consider the following models.

  ◇ **IID Model:** $T_1, T_2, \ldots, T_n$ IID with common unknown hazard rate function $\lambda(t)$. Observable vectors are

  $$(Z_1, \delta_1), (Z_2, \delta_2), \ldots, (Z_n, \delta_n)$$

  with

  $$\delta_i = 1 \Rightarrow T_i = Z_i$$

  $$\delta_i = 0 \Rightarrow T_i > Z_i.$$

  ◇ **Cox PH Model** (also Andersen and Gill Model): Let

  $$(T_1, X_1), (T_2, X_2), \ldots, (T_n, X_n)$$

  such that

  $$\lambda_{T|X}(t|X) = \lambda(t) \exp\{\beta^{\mathrm{t}} X\}$$

  $\lambda(\cdot)$ an unknown hazard rate function, and β a regression coefficient vector. The observable vectors are

  $$(Z_1, \delta_1, X_1), (Z_2, \delta_2, X_2), \ldots, (Z_n, \delta_n, X_n)$$

  with

  $$\delta_i = 1 \Rightarrow T_i = Z_i$$

  $$\delta_i = 0 \Rightarrow T_i > Z_i.$$

4. Problems, Issues, and Prior Works

- **Problem (Goodness-of-Fit):** Given

$$\{(Z_i, \delta_i), i = 1, 2, \ldots, n\}$$

in the IID model, or

$$\{(Z_i, \delta_i, X_i), i = 1, 2, \ldots, n\}$$

in the Cox model, decide whether

$$\lambda(\cdot) \in \mathcal{C} = \{\lambda_0(\cdot; \eta) : \eta \in \Upsilon\}$$

where η is a nuisance parameter vector.

  - $\diamond$ $\mathcal{C}$ could be: Exponential, Weibull, Pareto; or IFRA class.
  - $\diamond$ Importance: knowing $\lambda(\cdot) \in \mathcal{C}$ may improve inference procedures.
  - $\diamond$ Previous works on GOF problem: Akritas (88, JASA), Hjort (90, AS), Hollander and Peña (92, JASA), Li and Doss (93, AS), and others.
  - $\diamond$ How to generalize the Pearson-type statistic

$$\chi^2_P = \sum \frac{(O_j - \hat{E}_j)^2}{\hat{E}_j}?$$

  - $\diamond$ Difficulty in extending Pearson statistic: O_j's not computable.
  - $\diamond$ Optimality properties?

- **Problem (Model Validation):** Given $\{(Z_i, \delta_i), i = 1, 2, \ldots, n\}$ or $\{(Z_i, \delta_i, X_i), i = 1, 2, \ldots, n\}$, how to assess the viability of model assumptions?

  ◇ Unit Exponentiality Property (UEP)

$$T \sim \Lambda(\cdot) \Rightarrow \Lambda(T) \sim \mathrm{EXP}(1)$$

  ◇ IID model: If $\Lambda_0(\cdot)$ is the true hazard function, then with $R_i^0 = \Lambda_0(Z_i)$,

$$(R_1^0, \delta_1), (R_2^0, \delta_2), \ldots, (R_n^0, \delta_n)$$

  is a right-censored sample from EXP(1).

  ◇ Since $\Lambda_0(\cdot)$ is not known, the R_i^0's are estimated by R_i's with

$$R_i = \hat{\Lambda}(Z_i), \quad i = 1, 2, \ldots, n,$$

  $\hat{\Lambda}(\cdot)$ is an estimator of $\Lambda_0(\cdot)$ based on the (Z_i, δ_i)'s.

  ◇ **Idea:** (R_i, δ_i)'s *assumed to form an approximate right-censored sample from EXP(1),* so to validate model, test whether (R_i, δ_i)'s is a right-censored sample from EXP(1).

  ◇ **Question:** How good is the *approximation*, even in the limit???

◇ For Cox PH model, the analogous expressions for R_i^0 and R_i are:

$$R_i^0 = \Lambda_0(Z_i) \exp\{\beta X_i\}, \quad i = 1, 2, \ldots, n;$$

$$R_i = \hat{\Lambda}(Z_i) \exp\{\hat{\beta} X_i\}, \quad i = 1, 2, \ldots, n,$$

$\hat{\Lambda}(\cdot)$ is an estimator of $\Lambda_0(\cdot)$ [e.g., Aalen-Breslow estimator], while $\hat{\beta}$ is an estimator of β [partial likelihood MLE].

◇ R_i^0's are **true generalized residuals** (Cox and Snell (68, JRSS)); while R_i's are **estimated generalized residuals.**

◇ Generalized residuals are analogs of the linear model residuals:

"**(Observed Value)** minus **(Fitted Value)**."

◇ **Question:** What are the effects of substituting estimators for the unknown parameters??

5. Class of GOF Tests

- Convert observed data

$$(Z_1, \delta_1, X_1), (Z_2, \delta_2, X_2), \ldots, (Z_n, \delta_n, X_n)$$

into stochastic processes.

- For $i = 1, 2, \ldots, n$, and $t \geq 0$, let

$$N_i(t) = I\{Z_i \leq t, \delta_i = 1\} = \text{No. of uncensored failures;}$$

$$Y_i(t) = I\{Z_i \geq t\} = \text{No. at risk.}$$

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$$A_i(t; \lambda(\cdot), \beta) = \int_0^t Y_i(w)\lambda(w) \exp\{\beta^{\mathrm{t}} X_i\} \mathrm{d}w;$$

$$M_i(t; \lambda(\cdot), \beta) = N_i(t) - A_i(t; \lambda(\cdot), \beta).$$

- If $\lambda_0(\cdot)$ and β_0 are the *true* parameters,

$$M^0(t) = (M_1(t; \lambda_0(\cdot), \beta_0), \ldots, M_n(t; \lambda_0(\cdot), \beta_0))$$

are orthogonal sq-int zero-mean martingales with predictable quadratic variation processes

$$\langle M_i^0, M_i^0 \rangle(t) = A_i(t; \lambda_0(\cdot), \beta_0).$$

- **Problem:** Test

$$H_0 : \lambda(\cdot) \in \mathcal{C} = \{\lambda_0(\cdot; \eta) : \eta \in \Upsilon\} \quad \text{versus} \quad H_1 : \lambda(\cdot) \notin \mathcal{C}.$$

- **Idea:** If $\lambda_0(\cdot)$ is the true hazard rate function, then under H_0 there is some $\eta_0 \in \Upsilon$ such that

$$\lambda_0(\cdot) = \lambda_0(\cdot; \eta_0).$$

- Define

$$\kappa(t; \eta) = \log\left[\frac{\lambda_0(t)}{\lambda_0(t; \eta)}\right].$$

Denote by $\mathcal{K}$ the collection of such $\{\kappa(\cdot; \eta) : \eta \in \Upsilon\}$.

- Consider a basis set (e.g., trigonometric, polynomial, wavelet, etc.) for $\mathcal{K}$ given by

$$\{\psi_1(\cdot; \eta), \psi_2(\cdot; \eta), \ldots\}$$

so

$$\kappa(t; \eta) = \sum_{j=1}^{\infty} \theta_j \psi(t; \eta).$$

- For an appropriate order K (smoothing order), approximate $\kappa(\cdot; \eta)$ by

$$\kappa(t; \eta) = \sum_{j=1}^{K} \theta_j \psi(t; \eta).$$

- Equivalently,

$$\lambda_0(t) \approx \lambda_0(t; \eta) \exp\left\{\sum_{j=1}^{K} \theta_j \psi(t; \eta)\right\}.$$

- Define the class

$$\mathcal{C}_K = \left\{\lambda_K(\cdot; \theta, \eta) = \lambda_0(\cdot; \eta) \exp\left\{\sum_{j=1}^{K} \theta_j \psi_j(\cdot; \eta)\right\} : \quad \theta_K \in \Re^K; \eta \in \Upsilon\right\}.$$

- $H_0 \subset \mathcal{C}_K$.

- **Goodness-of-Fit Tests:** Score tests for

$$H_0 : \theta_K = 0, \beta \in \mathcal{B} \quad \text{versus} \quad H_1 : \theta_K \neq 0, \beta \in \mathcal{B}.$$

- Tests introduced in Peña (1998, JASA; 1998, AS).

- Since score tests, they possess some optimality properties.

- Score function for θ_K at $\theta_K = 0$:

$$Q(\eta, \beta) = \sum_{i=1}^{n} \int_0^\tau \Psi_K(\eta) \left\{ \mathrm{d}N_i - Y_i \lambda_0(\eta) \exp\{\beta^{\mathrm{t}} X_i\} \mathrm{d}t \right\}.$$

- **Not a statistic** since η and β are unknown.

- Need to plug-in estimators for η and β under the restriction $\theta_K = 0$.

- Estimate β by $\hat{\beta}$ which solves the estimating equation

$$S(\beta) \equiv \sum_{i=1}^{n} \int_0^\tau [X_i - E(\beta)] \mathrm{d}N_i = 0;$$

$$E(t, \beta) = \frac{S^{(1)}(t, \beta)}{S^{(0)}(t, \beta)};$$

$$S^{(m)}(t, \beta) = \sum_{i=1}^{n} X_i^{\otimes m} Y_i \exp\{\beta^{\mathrm{t}} X_i\}, \quad m = 0, 1, 2.$$

- Estimate η by $\hat{\eta}$ which solves the profile estimating equation

$$R(\eta, \hat{\beta}) \equiv \sum_{i=1}^{n} \int_0^\tau \rho(\eta) \left\{ \mathrm{d}N_i - Y_i \lambda_0(\eta) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \mathrm{d}t \right\} = 0;$$

$$\rho(t, \eta) = \frac{\partial}{\partial \eta} \log \lambda_0(t, \eta).$$

- Test statistic:

$$S_K = \frac{1}{n}\left\{\hat{Q}^{\mathrm{t}}\right\}\left\{\hat{\Xi}^{-1}\right\}\left\{\hat{Q}\right\},$$

with

$$\hat{Q} = Q(\hat{\eta},\hat{\beta}) = \sum_{i=1}^{n}\int_0^{\tau}\Psi_K(\hat{\eta})\left\{\mathrm{d}N_i - Y_i\lambda_0(\hat{\eta})\exp\{\hat{\beta}^{\mathrm{t}}X_i\}\mathrm{d}t\right\}$$

- $\hat{\Xi}$ is an estimator of the limiting covariance matrix of $\frac{1}{\sqrt{n}}\hat{Q}$.

- S_K ia a *function of the generalized residuals* $(R_i,\delta_i),\ldots,(R_n,\delta_n)$.

6. Asymptotics

- **Proposition:** If the parameters are known,

$$\frac{1}{\sqrt{n}}\begin{bmatrix} Q(\eta_0,\beta_0) \\ R(\eta_0,\beta_0) \\ S(\beta_0)\end{bmatrix} \xrightarrow{\mathrm{d}} N\left[\begin{pmatrix} 0 \\ 0 \\ 0\end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma_{21} & \Sigma_{22} & 0 \\ 0 & 0 & \Sigma_{33}\end{pmatrix}\right],$$

so

$$\frac{1}{\sqrt{n}}Q(\eta_0,\beta_0) \xrightarrow{\mathrm{d}} N(0,\Sigma_{11}).$$

- **Theorem:** With estimated parameters,

$$\frac{1}{\sqrt{n}}\hat{Q} \equiv \frac{1}{\sqrt{n}}Q(\hat{\eta},\hat{\beta}) \xrightarrow{\mathrm{d}} N_K(0,\ \Xi)$$

where

$$\Xi = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + (\Delta_1 - \Sigma_{12}\Sigma_{22}^{-1}\Delta_2)\Sigma_{33}^{-1}(\Delta_1 - \Sigma_{12}\Sigma_{22}^{-1}\Delta_2)^{\mathrm{t}}.$$

- Proofs rely on the martingale central limits theorem of Rebolledo.

7. Effects of the Plug-In Procedure

◇ From the covariance matrix Ξ

$$\Xi = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + (\Delta_1 - \Sigma_{12}\Sigma_{22}^{-1}\Delta_2)\Sigma_{33}^{-1}(\Delta_1 - \Sigma_{12}\Sigma_{22}^{-1}\Delta_2)^{\mathrm{t}},$$

plugging-in $(\hat{\eta}, \hat{\beta})$ for (η, β) to obtain the statistic $\hat{Q}$ has **no asymptotic effect** if

$$\Sigma_{12} = 0 \quad \text{and} \quad \Delta_1 = 0,$$

since Σ_{11} is the limiting covariance for $\frac{1}{\sqrt{n}}Q(\eta_0, \beta_0)$.

◇ Essence of "adaptiveness" [notion in semiparametrics; BKRW ('93); Cox and Reid ('87)]: it does not matter that the nuisance parameters (η, β) are unknown in $Q(\eta, \beta)$ since replacing them by their estimators does **not** make the asymptotic distribution of $Q(\hat{\eta}, \hat{\beta})$ different from $Q(\eta_0, \beta_0)$.

◇ $\Sigma_{12} = 0$ is an orthogonality condition between $\Psi_K(\eta_0)$ and $\rho(\eta_0)$.

◇ $\Delta_1 = 0$ is an orthogonality condition between $\rho(\eta_0)$ and $e(\eta_0, \beta_0)$.

◇ Orthogonality defined in an appropriate Hilbert space with inner product

$$\langle f, g \rangle = \int_0^\tau f g \nu_0(\mathrm{d}t),$$

on the class of square-integrable functions $L^2\{[0, \tau], \nu_0\}$, with

$$\nu_0(A) = \int_A s^{(0)}(\eta_0, \beta_0)\lambda_0(\eta_0)\mathrm{d}t.$$

◇ Can we always choose the Ψ_K to satisfy orthogonality conditions? Yes, via a Gram-Schmidt process but hard to implement!

◇ If orthogonality conditions are not satisfied, substituting $(\hat{\eta}, \hat{\beta})$ for (η_0, β_0) in $Q(\eta_0, \beta_0)$ impacts on the asymptotic distribution of $\hat{Q}$. *even though these estimators are consistent.*

◇ Effect contained in the last two terms in the expression for Ξ.

◇ Second term is the result of estimating η by $\hat{\eta}$; while the third term, which is an increase in the variance, is the effect of estimating β by the partial MLE $\hat{\beta}$.

◇ Estimating β by $\hat{\beta}$ leads to an increase in variance is because this estimator is less efficient than the full MLE of $\hat{\beta}$.

◇ Ignoring effect on variance could have **dire consequences** in the testing. If overall effect is a variance reduction, ignoring it may result in a highly conservative test and may lead into concluding model appropriateness when in fact model is inappropriate.

8. Form of the Test Procedure

- Recall:

$$\hat{Q} = Q(\hat{\eta}, \hat{\beta}) = \sum_{i=1}^{n} \int_0^{\tau} \Psi_K(\hat{\eta}) \left\{ \mathrm{d}N_i - Y_i \lambda_0(\hat{\eta}) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \mathrm{d}t \right\}.$$

- **Estimating Limiting Covariance Matrix, Ξ:** With

$$\hat{\Sigma} = \frac{1}{2n} \sum_{i=1}^{n} \int_0^{\tau} \begin{bmatrix} \Psi_K(\hat{\eta}) \\ \rho(\hat{\eta}) \\ X_i - E(\hat{\beta}) \end{bmatrix}^{\otimes 2} \left\{ \mathrm{d}N_i + Y_i \lambda_0(\hat{\eta}) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \mathrm{d}t \right\};$$

$$\hat{\Delta}_1 = \frac{1}{2n} \sum_{i=1}^{n} \int_0^{\tau} \Psi_K(\hat{\eta}) E(\hat{\beta})^{\mathrm{t}} \left\{ \mathrm{d}N_i + Y_i \lambda_0(\hat{\eta}) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \mathrm{d}t \right\};$$

and

$$\hat{\Delta}_2 = \frac{1}{2n} \sum_{i=1}^{n} \int_0^{\tau} \rho(\hat{\eta}) E(\hat{\beta})^{\mathrm{t}} \left\{ \mathrm{d}N_i + Y_i \lambda_0(\hat{\eta}) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \mathrm{d}t \right\};$$

an estimator of Ξ is

$$\hat{\Xi} = \hat{\Sigma}_{11} - \hat{\Sigma}_{12} \hat{\Sigma}_{22}^{-1} \hat{\Sigma}_{21} + (\hat{\Delta}_1 - \hat{\Sigma}_{12} \hat{\Sigma}_{22}^{-1} \hat{\Delta}_2) \hat{\Sigma}_{33}^{-1} (\hat{\Delta}_1 - \hat{\Sigma}_{12} \hat{\Sigma}_{22}^{-1} \hat{\Delta}_2)^{\mathrm{t}}.$$

- **Form of Asymptotic α-Level Test:**

Reject $H_0 : \lambda(\cdot) \in \mathcal{C}; \beta \in \mathcal{B}$ whenever

$$S_K = \frac{1}{n} \left\{ \hat{Q}^{\mathrm{t}} \right\} \left\{ \hat{\Xi}^{-} \right\} \left\{ \hat{Q} \right\} \geq \chi^2_{K^*;\alpha},$$

$\hat{\Xi}^{-}$ = Moore-Penrose generalized inverse of $\hat{\Xi}$ and $K^* = \mathrm{rank}(\hat{\Xi})$.

9. Choices for Ψ_K

- Partition $[0, \tau]$: $0 = a_0 < a_1 < ... < a_{K-1} < a_K = \tau$
- Let

$$\Psi_K(t) = \left[I_{[0,a_1]}(t), I_{(a_1,a_2]}(t), ..., I_{(a_{K-1},\tau]}(t)\right]^{\mathrm{t}}.$$

- Then

$$\hat{Q} = [O_1 - E_1, O_2 - E_2, \ldots, O_K - E_K]^{\mathrm{t}}$$

where

$$O_j = \sum_{i=1}^{n} \int_{a_{j-1}}^{a_j} \mathrm{d}N_i(t)$$

$$E_j = \sum_{i=1}^{n} \int_{a_{j-1}}^{a_j} Y_i(t) \exp\{\hat{\beta}^{\mathrm{t}} X_i\} \lambda_0(t; \hat{\eta}) \mathrm{d}t.$$

- O_j's are observed frequencies.
- E_j's are *estimated* dynamic expected frequencies.
- Extends Pearson's statistic, but 'counts are in a dynamic fashion.'
- Resulting test statistic **not** of form

$$\sum_{j=1}^{K} \frac{(O_j - E_j)^2}{E_j}$$

because correction terms in variance destroys diagonal nature of covariance matrix.

- If adjustments are ignored, distribution is **not** chi-squared.
- Procedure extends Akritas (1988) and Hjort (1990).

- **Example:** Consider the no-covariate (so $X_i = 0$), and with $\mathcal{C} = \{\lambda_0(t;\eta) = \eta\}$ (Exponential distribution).

- For $j = 1, 2, \ldots, K$,

$$O_j = \sum_{i=1}^{n} \int_{a_{j-1}}^{a_j} \mathrm{d}N_i(t);$$

$$E_j = \hat{\eta} \sum_{i=1}^{n} \int_{a_{j-1}}^{a_j} Y_i(t)\mathrm{d}t;$$

$$\hat{\eta} = \frac{\sum_{i=1}^{n} \int_0^{\tau} \mathrm{d}N_i(t)}{\sum_{i=1}^{n} \int_0^{\tau} Y_i(t)\mathrm{d}t} = \frac{\#\text{ of events}}{\text{total exposure}}.$$

- Resulting test statistic becomes

$$S_K = E_{\bullet} \left[\mathbf{p} - \hat{\pi}\right]^{\mathrm{t}} \left[\mathrm{Dg}(\hat{\pi}) - \hat{\pi}\hat{\pi}^{\mathrm{t}}\right]^{-} \left[\mathbf{p} - \hat{\pi}\right],$$

where

$$E_{\bullet} = \sum_{j=1}^{K} E_j;$$

$$\mathbf{p} = \frac{1}{E_{\bullet}}(O_1, O_2, \ldots, O_K)^{\mathrm{t}};$$

$$\hat{\pi} = \frac{1}{E_{\bullet}}(E_1, E_2, \ldots, E_K)^{\mathrm{t}}.$$

- $E_{\bullet} \neq n$.

- Appropriate df for statistic is $K - 1$ since $\mathbf{1}^{\mathrm{t}}\hat{\pi} = 1$.

- **Polynomial-type** of basis:

$$\Psi_K(t) = \left[1, \Lambda_0(t;\eta), \Lambda_0(t;\eta)^2, \ldots, \Lambda_0(t;\eta)^{K-1}\right]^{\mathrm{t}}.$$

- Resulting $\hat{Q}$ vector has components

$$\hat{Q}_j = \sum_{i=1}^{n} \exp\{-(j-1)\hat{\beta}X_i\} \int_0^{\tau} w^{j-1} \left\{\mathrm{d}N_i^R(w) - Y_i^R(w)\mathrm{d}w\right\},$$

where

$$N_i^R(w) = I\{R_i \le w; \delta_i = 1\};$$

$$Y_i^R(w) = I\{R_i \ge w\};$$

$$R_i = \Lambda_0(Z_i;\hat{\eta})\exp\{\hat{\beta}X_i\}.$$

- N_i^R's and Y_i^R's are generalized residual processes.

- Generalizes Hyde's (mid '70's).

- **Total-Time-On-Test Specification:** For $K = 1$, define

$$\hat{\tau}^0 = \Lambda_0(\tau;\hat{\eta});$$

$$N_0^R(t) = N[\Lambda_0^{-1}(t;\hat{\eta})];$$

$$Y_0^R(t) = Y[\Lambda_0^{-1}(t;\hat{\eta})];$$

$$R_0^R(t) = \int_0^t Y_0^R(s)\mathrm{d}s.$$

- Let

$$\psi_1^0(t) = \frac{R_0^R(t)}{R_0^R(\tau^0)} - \frac{1}{2}.$$

- **Resulting Test:** Reject H_0 whenever

$$S_1 = \left[\frac{N_0^R(\hat{\tau}^0)}{R_0^R(\hat{\tau}^0)}\right] \frac{[Q^R(\hat{\tau}^0)]^2}{[1 - 12\hat{\Delta}(\hat{\tau}^0)]} \geq \chi^2_{1;\alpha},$$

where

$$\hat{Q} = \sqrt{12 N_0^R(\hat{\tau}^0)} \left\{ \frac{1}{N_0^R(\hat{\tau}^0)} \int_0^{\hat{\tau}^0} \frac{R_0^R(t)}{R_0^R(\hat{\tau}^0)} \mathrm{d}N_0^R(t) - \frac{1}{2} \right\};$$

$$\hat{\Delta}(\hat{\tau}^0) = \frac{\hat{\gamma}_1 \hat{\mathbf{\Psi}}^{-1} \hat{\gamma}_1^{\mathrm{t}}}{n^{-1} R_0^R(\hat{\tau}^0)}.$$

- In notation usable for "teaching purposes,"

$$\hat{Q} = \sqrt{12 n^*} \left[\frac{1}{n^*} \sum_{i=1}^{n} \delta_{(i)} \frac{W_{(i)}}{W_{(n)}} - \frac{1}{2} \right]^2,$$

where

$$n^* = \sum_{i=1}^{n} \delta_i = \text{number of observed failures};$$

$$W_{(i)} = \sum_{j=1}^{i} (n - j + 1)(R_{(j)} - R_{(j-1)});$$

$$R_{(j)} = j\text{th smallest generalized residual.}$$

- $Q^R(\hat{\tau}^0)$: generalizes the **normalized spacings test** *applied to the generalized residual processes* (N_0^R, Y_0^R).

- Good for detecting IFR alternatives.

- **Bonus:** Able to show that this normalized spacings test is a score test!

- $[1 - 12\hat{\Delta}(\hat{\tau}^0)]^{-1}$ represents variance adjustment due to the estimation of nuisance parameter η by $\hat{\eta}$.

- If $\mathcal{C}$ is the constant hazard class, **no** correction is needed *even* with censored data. **Adaptiveness rules!**

- If $\mathcal{C}$ is the two-parameter Weibull, correction term is not ignorable. For **complete data**,

$$[1 - 12\Delta(\infty)]^{-1} = \left[1 - \frac{18(\log 2)^2}{\pi^2}\right]^{-1} = 8.0802....$$

10. Levels of Tests for Different Smoothing Order, K

- **Polynomial:** $\Psi_K(t;\eta) = \big(1, \Lambda_0(t;\eta), \ldots, \Lambda_0(t;\eta)^{K-1}\big)^{\mathrm{t}}$
- $K \in \{1,2,3,4\}$; censoring proportion $\in \{25\%, 50\%\}$ and true failure time model were exponential and 2-Weibull. # of Reps = 2000

Null Dist.		Exponential(η)			
Parameters		$\eta = 2$		$\eta = 5$	
% Uncensored		75%	50%	75%	50%
Level		5%	5%	5%	5%
n	K				
50	2	4.65	6.65	5.00	5.60
	3	5.45	5.50	4.35	4.95
	4	6.55	5.50	5.10	5.85
	5	6.40	5.00	5.20	4.20
100	2	4.90	4.75	4.45	4.35
	3	4.55	4.35	4.65	4.25
	4	5.70	5.30	5.10	4.80
	5	5.75	4.90	5.30	4.75

Null Dist.		Weibull(α, η)			
Parameters		$(\alpha,\eta) = (2,1)$		$(\alpha,\eta) = (3,2)$	
% Uncensored		75%	50%	75%	50%
Level		5%	5%	5%	5%
n	K				
50	2	4.30	4.80	4.60	6.20
	3	5.40	5.15	6.30	5.70
	4	4.80	4.80	6.10	5.30
	5	5.25	3.45	6.70	4.60
100	2	3.90	4.95	4.20	4.80
	3	5.75	4.65	5.15	5.25
	4	5.55	4.15	5.00	4.30
	5	6.00	4.65	5.80	5.30

11. Power Function of Tests

Legend: Solid line ($K = 2$); Dots ($K = 3$); Short dashes ($K = 4$); Long dashes ($K = 5$)

Figure 2: Simulated Powers for Null: **Exponential** vs. Alt: **2-Weibull**

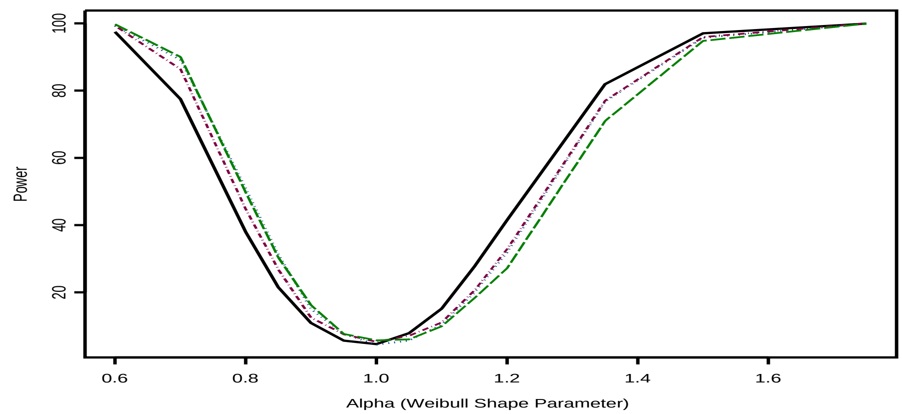


Figure 3: Simulated Powers for Null: **Exponential** vs. Alt: **2-Gamma**

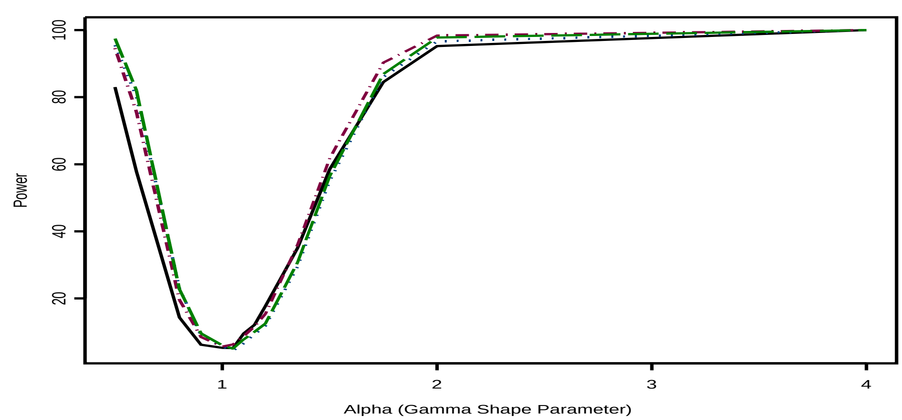
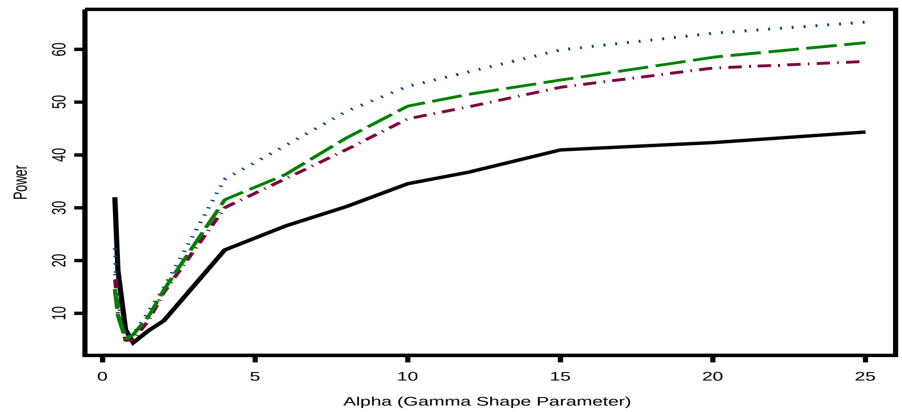


Figure 4: Simulated Powers for Null: **2-Weibull** vs. Alt: **2-Gamma**

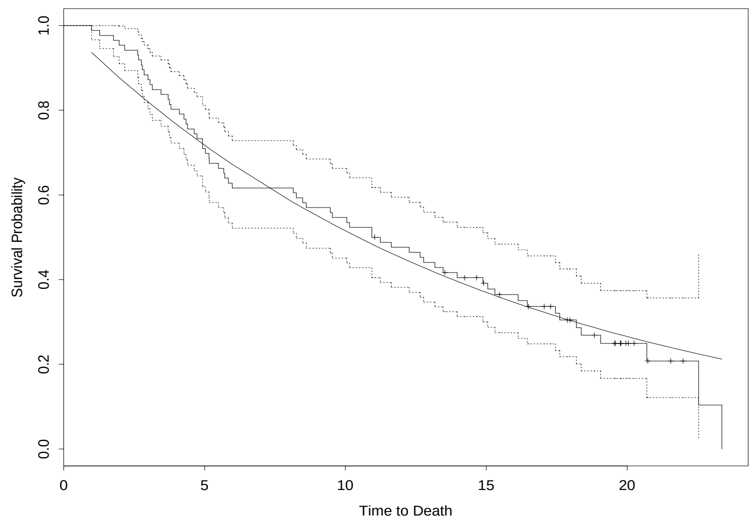


Some Observations:

- Appropriate smoothing order K depends on alternative considered.
- Not always necessary to have large K!
- Tests based on S_3 and S_4 could serve as omnibus tests, at least for the models in these simulations.
- Calls for a formal method to dynamically determine K.

12. Back to the Lung Cancer Data

- Product-Limit Estimator (PLE) of survival curve together with confidence band, and the best fitting exponential survival curve.



- **Testing Exponentiality:** Values of test statistics using the polynomial-type specification, together with their p-values are:

$$S_2 = 1.92(p = .1661); \quad S_3 = 1.94(p = .3788);$$

$$S_4 = 7.56(p = .0561); \quad S_5 = 12.85(p = .0121).$$

- Close to a constant function, but with high frequency terms.

- **Testing Two-Parameter Weibull:** Value of S_3, together with its p-value, is

$$S_3 = 8.35 \quad (p = .0153)$$

Values of S_4 and S_5 both indicate rejection of two-parameter Weibull model.

13. Concluding Remarks

- Presented a formal approach to develop GOF tests with censored data.
- Able to see effects of estimating nuisance parameters: (**Don't Ignore!**)
- Potential problems when using generalized residuals in model validation: (**Intuitive considerations may Fail!**)
- Promising possibility: **Wavelets as Basis??!**
- Discrete hazard modeling. Almost finished with this.
- Open problems:
- How about a detailed comparison with existing tests: KS, CVM, etc.
- How to determine smoothing order K adaptively?
- What if $\mathcal{C}$ is a nonparametric life distribution class such as the IFRA?