# Goodness-of-Fit Testing with <br> Discrete Censored Data 

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## 1. GOF Testing for Continuous/Discrete Data

- GOF testing is a major research area in Statistics.
- MathSciNet search for "goodness of fit": at least 650 hits.
- $T_{1}, T_{2}, \ldots, T_{n}$ IID from a distribution function $F$.
- (Simple Case) $H_{0}: F=F_{0}$ versus $H_{1}: F \neq F_{0}$.
- (Composite Case) $H_{0}: F \in \mathcal{C}=\{F(\cdot ; \theta): \theta \in \Theta\}$ versus $H_{1}: F \notin \mathcal{C}$.
- Pearson's (1900) statistic for simple null case:

$$
\chi^{2}=\sum_{j=1}^{K} \frac{\left(O_{j}-E_{j}\right)^{2}}{E_{j}}
$$

- Appealing, simple, and requires only chi-squared critical values.
- Other tests: KS, CVM, Neyman's smooth tests, Khamaladze, Rao-Robson, review in Stephens; cf., D'Agostino and Stephens.
- Most of these tests require: F is continuous. Reason? nice distributional results.
- But, discrete data are also ubiquitous in practical settings.
- Nature of event of interest (e.g., count data), limitations in the measurement process (e.g., interval or life-table data), quantum theory!
- Pearson's procedure still applicable for discrete data.
- Cressie and Read (84) power divergence tests; Kulperger and Singh (84) $\chi^{2}$-type tests; Choulakian, Lockhart and Stephens' (94) test for discrete uniform; Spinelli and Stephens (97) CVMtest Poisson distribution.
- Best and Rayner $(89,99)$ Neyman's smooth tests for geometric and Poisson; Eubank (97) Neyman's smooth test for multinomial data.
- Kocherlakota and Kocherlakota (86), Rueda, Perez-Abreau and O'Reilly (91), Baringhaus and Henze (92), and Nakamura and Perez-Abreau (93) examined gof tests for discrete data using the empirical probability generating function.
- Empirical distribution-based methods also considered for discrete models, cf., Henze (96) and Klar (99).
- However, these papers assume that $T_{1}, T_{2}, \ldots, T_{n}$ are completely observed.


## 2. Right-Censored Failure-Time Data

- Biomedical, public health, and reliability settings: Interest is a failure time. Censoring occurs due to time and resource contraints, withdrawal from the study, loss to follow-up, etc.
- Several papers have addressed this GOF problem with the aim of extending to censored data procedures for complete data.
- Among these papers are Koziol and Green (76), Hyde (77), Hollander and Proschan (79), Nair (81, 82, 84), Gatsonis, Hsieh and Korwar (85), Habib and Thomas (86), Akritas (88), Hjort (90), Hollander and Peña (92), Li and Doss (93), and Kim (93).
- Particular goal: extend Pearson's test. Difficulty in extension is that exact number of failures in a partition not observable.
- Neyman's smooth tests extended Gray and Pierce (85) for right-censored data. Approach parallels Neyman (37) where density function is embedded in a wider class.
- Hazard-based extension of smooth tests with continuous data made in Peña (98ab). Formulation adapts naturally to censored data, and allowed martingale theory to be used.
- Except for Hyde (77), and surprisingly, the GOF problem with right-censored discrete failure times have not been investigated extensively.


## - Goals of this talk:

1. To describe a hazard-based formulation for generating a general class of gof tests with discrete right-censored data. This formulation is the discrete case analog of the hazardbased formulation for continuous data in Peña (98ab).
2. To present a general omnibus gof test with good power against a wide variety of alternatives, together with some directional tests which focuses on specific alternatives.
3. Discuss analogs of Pearson's gof test.
4. To illustrate the class of tests by providing tests for the geometric distribution.
5. Present simulation results pertaining to the achieved levels and powers of the different tests.

- $T \in \mathcal{A}=\left\{a_{1}, a_{2}, a_{3}, \ldots\right\}$ is a discrete failure-time variable with df $F(t)=\mathbf{P}\{T \leq t\}$. Assume $a_{i}<a_{i+1}$.
- $\lambda_{j}=\mathbf{P}\left\{T=a_{j} \mid T \geq a_{j}\right\}, j=1,2, \ldots$ are the hazard rates.
- $\left\{\lambda_{1}^{0}, \lambda_{2}^{0}, \ldots, \lambda_{j}^{0}, \ldots\right\}$ are the hazard rates for $F_{0}$.
- Let $J$ be a fixed positive integer, with $\left[0, a_{J}\right]$ being the observation period for the study.
- GOF problem:

$$
\begin{aligned}
& H_{0}: \lambda_{j}=\lambda_{j}^{0} \text { for all } j \in\{1,2, \ldots, J\} ; \\
& H_{1}: \lambda_{j} \neq \lambda_{j}^{0} \text { for some } j \in\{1,2, \ldots, J\} .
\end{aligned}
$$

- Failure Times: $T_{1}, T_{2}, \ldots, T_{n}$ IID $F$.
- Censoring Times: $C_{1}, C_{2}, \ldots, C_{n}$ with $C_{i} \in \mathcal{A}$ and for each $i=1,2, \ldots, n$,

$$
\mathbf{P}\left\{T_{i}=a_{j} \mid T_{i} \geq a_{j}, C_{i} \geq a_{j}\right\}=\mathbf{P}\left\{T_{i}=a_{j} \mid T_{i} \geq a_{j}\right\}=\lambda_{j}
$$

for $j=1,2, \ldots$.

- Independent censoring condition.
- Right-censored data: With $Z_{i}=\min \left(T_{i}, C_{i}\right)$ and $\delta_{i}=$ $I\left\{T_{i} \leq C_{i}\right\}$,

$$
\left(Z_{1}, \delta_{1}\right),\left(Z_{2}, \delta_{2}\right), \ldots,\left(Z_{n}, \delta_{n}\right)
$$

## 4. Developing the GOF Tests

- Goal: To test $H_{0}$ vs $H_{1}$ based on right-censored data.
- Define the true and null hazard odds

$$
\rho_{j}=\frac{\lambda_{j}}{1-\lambda_{j}} \quad \text { and } \quad \rho_{j}^{0}=\frac{\lambda_{j}^{0}}{1-\lambda_{j}^{0}} .
$$

- GOF problem: Test $H_{0}: \rho_{j}=\rho_{j}^{0}, j=1,2, \ldots, J$, versus $H_{1}: \rho_{j} \neq \rho_{j}^{0}$ for some $j \in\{1,2, \ldots, J\}$.
- Let $p$ be a pre-specified positive integer, called the smoothing order.
- Let $\boldsymbol{\Psi}$ be a, possibly random, $p \times J$ matrix:

$$
\boldsymbol{\Psi}=\left(\boldsymbol{\Psi}_{1}, \boldsymbol{\Psi}_{2}, \ldots, \boldsymbol{\Psi}_{J}\right)=\left[\begin{array}{c}
\psi_{1} \\
\psi_{\mathbf{2}} \\
\vdots \\
\psi_{\mathbf{p}}
\end{array}\right]
$$

- $\Psi_{j}, j=1, \ldots, J$, are $p \times 1$ vectors; while $\psi_{k}, k=1, \ldots, p$, are $1 \times J$ vectors. Assume $\psi_{1}, \ldots, \psi_{p}$ are linearly independent.
- Embed $\left(\rho_{1}^{0}, \rho_{2}^{0}, \ldots, \rho_{J}^{0}\right)$ in the class

$$
\mathcal{C}_{p}=\left\{\left(\rho_{1}(\theta), \rho_{2}(\theta), \ldots, \rho_{J}(\theta)\right): \theta=\left(\theta_{1}, \theta_{2}, \ldots, \theta_{p}\right)^{\mathrm{t}} \in \Re^{p}\right\}
$$

where

$$
\rho_{j}(\theta)=\rho_{j}^{0} \exp \left\{\theta^{\mathrm{t}} \boldsymbol{\Psi}_{j}\right\} \quad \text { for } \quad j=1,2, \ldots, J
$$

- $\lambda_{j}(\theta)=\rho_{j}^{0} \exp \left\{\theta^{\mathrm{t}} \boldsymbol{\Psi}_{j}\right\} /\left\{1+\rho_{j}^{0} \exp \left\{\theta^{\mathrm{t}} \boldsymbol{\Psi}_{j}\right\}\right\}, j=1, \ldots, J$.
- Amounts to assuming that logarithms of the hazard odds ratios satisfy

$$
\log \left\{\frac{\rho_{j}(\theta)}{\rho_{j}^{0}}\right\}=\theta^{\mathrm{t}} \mathbf{\Psi}_{j}=\sum_{k=1}^{p} \theta_{k} \Psi_{k j}, \quad j=1,2, \ldots, J
$$

- Logit-based formulation.
- Justification: Think of the mapping

$$
j \in\{1,2, \ldots, J\} \mapsto h_{j} \equiv \log \left\{\rho_{j} / \rho_{j}^{0}\right\}
$$

as having range $\mathcal{H}$, a subspace of $\Re^{J}$. Let $\left\{\psi_{1}^{\mathrm{t}}, \psi_{2}^{\mathrm{t}}, \ldots, \psi_{J}^{\mathrm{t}}\right\}$ be $J \times 1$ vectors forming a basis of $\mathcal{H}$ so

$$
\left(\log \left\{\frac{\rho_{j}}{\rho_{j}^{0}}\right\}_{j=1,2, \ldots, J}\right)^{\mathrm{t}}=\sum_{k=1}^{J} \theta_{k} \psi_{k}^{\mathrm{t}} .
$$

- If $\psi_{j}$ 's are ordered according to some criterion (e.g., frequency), then truncate summation up to order $p$ to obtain

$$
\left(\log \left\{\frac{\rho_{j}}{\rho_{j}^{0}}\right\}_{j=1,2, \ldots, J, J}\right)^{\mathrm{t}} \approx \sum_{k=1}^{p} \theta_{k} \psi_{k}^{\mathrm{t}} .
$$

- Solve for $\rho_{j}$ to obtain the embedding class.
- GOF problem reduces to $H_{0}^{*}: \theta=\mathbf{0}$ versus $H_{1}^{*}: \theta \neq \mathbf{0}$.
- Members of proposed class of gof tests are score tests arising from varying $\boldsymbol{\Psi}$ and $p$.
- For $j=1,2, \ldots$, let

$$
\begin{gathered}
O_{j}=\sum_{i=1}^{n} I\left\{Z_{i}=a_{j}, \delta_{i}=1\right\} ; \\
R_{j}=\sum_{i=1}^{n} I\left\{Z_{i} \geq a_{j}\right\}
\end{gathered}
$$

- Theorem 4.1: Under the independent censoring condition, the relevant partial likelihood of $\theta$, given $\left(Z_{i}, \delta_{i}\right), i=$ $1,2, \ldots, n$, is

$$
L_{1}(\theta)=\prod_{j=1}^{J}\left[\lambda_{j}(\theta)\right]^{O_{j}}\left[1-\lambda_{j}(\theta)\right]^{R_{j}-O_{j}}=\prod_{j=1}^{J} \frac{\left[\rho_{j}(\theta)\right]^{O_{j}}}{\left[1+\rho_{j}(\theta)\right]^{R_{j}}}
$$

- From this partial likelihood, the score vector and observed information matrix are obtained.
- Theorem 4.2: Under same conditions, the score vector and observed information matrix associated with $\theta \mapsto$ $L_{1}(\theta)$ under $H_{0}^{*}: \theta=\mathbf{0}$, are

$$
\begin{array}{r}
\mathbf{U}_{0}^{*}(\boldsymbol{\Psi})=\sum_{j=1}^{J} \boldsymbol{\Psi}_{j}\left[O_{j}-R_{j} \lambda_{j}^{0}\right]=\boldsymbol{\Psi}\left(\mathbf{O}-\mathbf{E}_{0}\right) ; \\
\mathbf{I}_{0}^{*}(\mathbf{\Psi})=\sum_{j=1}^{J} \boldsymbol{\Psi}_{j}^{\otimes 2} R_{j} \lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right)=\boldsymbol{\Psi} \mathbf{V}_{0} \boldsymbol{\Psi}^{\mathrm{t}}, \\
\text { with } \mathbf{V}_{0}=\operatorname{Diag}\left(R_{j} \lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right), j=1, \ldots, J\right) .
\end{array}
$$

## 5. Asymptotics

- Define, for $i=1,2, \ldots$ and $j=1,2, \ldots, J$,

$$
\begin{gathered}
V_{i j}=I\left\{Z_{i}=a_{j}, \delta_{i}=1\right\}=I\left\{T_{i}=a_{j}, C_{i} \geq a_{j}\right\} \\
W_{i j}=I\left\{Z_{i} \geq a_{j}\right\}=I\left\{T_{i} \geq a_{j}, C_{i} \geq a_{j}\right\} \\
\mathcal{F}_{j}=\bigvee_{i=1}^{n} \sigma\left\{W_{i 1}, V_{i 1}, W_{i 2}, \ldots, V_{i j}, W_{i j+1}\right\}
\end{gathered}
$$

- Theorem 5.1: If $\boldsymbol{\Psi}$ is such that $\boldsymbol{\Psi}_{j}$ is $\mathcal{F}_{j-1}$-measurable, $p$ does not change with $n$, and there exists a $p \times p$ positive definite matrix $\Sigma_{0}$ such that, as $n \rightarrow \infty$,
(i) $\frac{1}{n} \mathbf{I}_{0}^{*}=\frac{1}{n} \boldsymbol{\Psi} \mathbf{V}_{0} \boldsymbol{\Psi}^{\mathrm{t}} \xrightarrow{\mathrm{pr}} \boldsymbol{\Sigma}_{0}$;
(ii) $\max _{1 \leq j \leq J} \operatorname{trace}\left\{\left(\boldsymbol{\Psi} \mathbf{V}_{0} \boldsymbol{\Psi}^{\mathrm{t}}\right)^{-1}\left(\boldsymbol{\Psi}_{j} V_{j j}^{0} \boldsymbol{\Psi}_{j}^{\mathrm{t}}\right)\right\} \xrightarrow{\mathrm{pr}} 0$;
(iii) $\max _{1 \leq j \leq J}\left\|\Psi_{j}\right\|^{2}=O_{p}(1)$,
then, under $H_{0}$,

$$
\frac{1}{\sqrt{n}} \mathbf{U}_{0}^{*}=\frac{1}{\sqrt{n}} \mathbf{\Psi}(\mathbf{O}-\mathbf{E}) \xrightarrow{\mathrm{d}} N_{p}\left(\mathbf{0}, \boldsymbol{\Sigma}_{0}\right) .
$$

Therefore, with $k=\operatorname{rank}\left(\boldsymbol{\Sigma}_{0}\right)$,

$$
S^{2}(\boldsymbol{\Psi})=\left(\mathbf{O}-\mathbf{E}_{0}\right)^{\mathrm{t}} \boldsymbol{\Psi}^{\mathrm{t}}\left(\boldsymbol{\Psi} \mathbf{V}_{0} \boldsymbol{\Psi}^{\mathrm{t}}\right)^{-} \boldsymbol{\Psi}\left(\mathbf{O}-\mathbf{E}_{0}\right) \xrightarrow{\mathrm{d}} \chi_{k}^{2}
$$

- Form of the Smooth GOF Test: Reject $H_{0}$ whenever

$$
S^{2}(\boldsymbol{\Psi})=\left(\mathbf{O}-\mathbf{E}_{0}\right)^{\mathrm{t}} \mathbf{\Psi}^{\mathrm{t}}\left(\boldsymbol{\Psi} \mathbf{V}_{0} \mathbf{\Psi}^{\mathrm{t}}\right)^{-} \boldsymbol{\Psi}\left(\mathbf{O}-\mathbf{E}_{0}\right) \geq \chi_{k^{*} ; \alpha}^{2}
$$

where $k^{*}$ is the rank of $\mathbf{I}_{0}^{*}=\boldsymbol{\Psi} \mathbf{V}_{0} \boldsymbol{\Psi}^{\mathrm{t}}$.

## 6. Some Special Cases

- Cases with $p=1$.
- Let $\psi_{1}=\mathbf{1}_{\mathcal{J}}^{\mathrm{t}}=(1,1, \ldots, 1)$. Then

$$
S^{2}\left(\psi_{1}\right)=\frac{\left[\sum_{j=1}^{J}\left(O_{j}-E_{j}^{0}\right)\right]^{2}}{\sum_{j=1}^{J} R_{j} \lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right)}=\left[\frac{O_{\bullet}-E_{\bullet}^{0}}{\sqrt{V_{\bullet}^{0}}}\right]^{2},
$$

where $O_{\bullet}=\sum_{j=1}^{J} O_{j}, E_{\bullet}^{0}=\sum_{j=1}^{J} E_{j}^{0}$, and $V_{\bullet}^{0}=\sum_{j=1}^{J} V_{j j}^{0}=$ $\sum_{j=1}^{J} R_{j} \lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right)$.

- Hyde's (1977) gof test statistic for discrete censored data.
- Let $\psi_{2}=\sqrt{n}\left(\frac{I\left\{V_{1}^{0}>0\right\}}{\sqrt{V_{11}^{0}}}, \frac{I\left\{V_{2>}^{0}>0\right\}}{\sqrt{V_{22}^{0}}}, \ldots, \frac{I\left\{V_{J_{J}^{0}}^{0}>0\right\}}{\sqrt{V_{J J}^{0}}}\right)^{\mathrm{t}}$. Leads to

$$
S^{2}\left(\psi_{2}\right)=\left\{\frac{1}{\sqrt{J^{*}}} \sum_{j=1}^{J} \sqrt{R_{j}}\left[\frac{\hat{\lambda}_{j}-\lambda_{j}^{0}}{\sqrt{\lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right)}}\right]\right\}^{2},
$$

where $\hat{\lambda}_{j}=O_{j} / R_{j}$ and $J^{*}=\sum_{j=1}^{J} I\left\{V_{j j}^{0}>0\right\}$.

- Weighted 'binomial-type' statistics.
- For a nonrandom $\gamma \in \Re$, let

$$
\psi_{3}^{\gamma}=\left[\left(\frac{R_{1}}{n}\right)^{\gamma} I\left\{R_{1}>0\right\}, \ldots,\left(\frac{R_{J}}{n}\right)^{\gamma} I\left\{R_{J}>0\right\}\right]^{\mathrm{t}} .
$$

Leads to

$$
S^{2}\left(\psi_{3}^{\gamma}\right)=\left\{\frac{\sum_{j=1}^{J} I\left\{R_{j}>0\right\} R_{j}^{1+\gamma}\left(\hat{\lambda}_{j}-\lambda_{j}^{0}\right)}{\sqrt{\sum_{j=1}^{J} I\left\{R_{j}>0\right\} R_{j}^{1+2 \gamma} \lambda_{j}^{0}\left(1-\lambda_{j}^{0}\right)}}\right\}^{2} .
$$

- Cases with $p>1$.
- Let $p \in \mathcal{Z}_{+}$with $p \leq J$, and let $A_{1}, A_{2}, \ldots, A_{p}$ with $A_{i} \neq$ $\emptyset, i=1,2, \ldots, p$, be a partition of $\mathcal{J}$. Define $\boldsymbol{\Psi}_{5}$ to be the (nonrandom) $p \times J$ matrix

$$
\mathbf{\Psi}_{5}=\left[\mathbf{1}_{A_{1}}, \mathbf{1}_{A_{2}}, \ldots, \mathbf{1}_{A_{p}}\right]^{\mathrm{t}} .
$$

- For $A \subseteq \mathcal{J}$, let $O \bullet(A)=\Sigma_{j \in A} O_{j}, E_{\bullet}^{0}(A)=\Sigma_{j \in A} E_{j}^{0}$, and $V_{\bullet}^{0}(A)=\Sigma_{j \in A} V_{j j}^{0}=\Sigma_{j \in A} E_{j}^{0}\left(1-\lambda_{j}^{0}\right)$.
- $\Psi_{5}$ induces

$$
S^{2}\left(\boldsymbol{\Psi}_{5}\right)=\sum_{i=1}^{p} \frac{\left[O_{\bullet}\left(A_{i}\right)-E_{\bullet}^{0}\left(A_{i}\right)\right]^{2}}{V_{\bullet}^{0}\left(A_{i}\right)} .
$$

- Analogous to Pearson's except that divisors are $V_{\bullet}^{0}\left(A_{i}\right)$ 's instead of $E_{\bullet}^{0}\left(A_{i}\right)$ 's.
- $E_{\bullet}^{0}\left(A_{i}\right)$ 's are dynamic expected frequencies.
- An interesting special case of $\boldsymbol{\Psi}_{5}$ is to take $p=J$ and $A_{i}=$ $\{i\}, i=1,2, \ldots, J$ so $\boldsymbol{\Psi}_{5}=\mathbf{I}_{J}$. Test statistic becomes

$$
S^{2}\left(\mathbf{I}_{J}\right)=\sum_{j=1}^{J} \frac{\left(O_{j}-E_{j}^{0}\right)^{2}}{E_{j}^{0}\left(1-\lambda_{j}^{0}\right)} .
$$

- Intuitive appeal because of its simplicity and similarity with the Pearson statistic. But, performed poorly in simulations!
- A specification with a random smoothing matrix.
- Let $\mathbf{R}=\left(R_{1}, R_{2}, \ldots, R_{J}\right)^{\mathrm{t}}$ and, for $k \in \mathcal{Z}_{+}$,

$$
\mathbf{R}^{k} \equiv\left(R_{1}^{k}, R_{2}^{k}, \ldots, R_{J}^{k}\right)^{\mathrm{t}}
$$

- Given $p \in \mathcal{Z}_{+}$with $p \leq J$, define the $p \times J$ random matrix $\boldsymbol{\Psi}_{6}^{p}$ via

$$
\mathbf{\Psi}_{6}^{p}=\left[\left(\frac{\mathbf{R}}{n}\right)^{0},\left(\frac{\mathbf{R}}{n}\right)^{1}, \ldots,\left(\frac{\mathbf{R}}{n}\right)^{p-1}\right]^{\mathrm{t}}
$$

- The $j$ th column of $\boldsymbol{\Psi}_{6}^{p}$ is

$$
\boldsymbol{\Psi}_{j}^{p}=\left[1,\left(\frac{R_{j}}{n}\right),\left(\frac{R_{j}}{n}\right)^{2}, \ldots,\left(\frac{R_{j}}{n}\right)^{p-1}\right]^{\mathrm{t}}
$$

- Serves as random polynomial basis vectors!
- For $i, i_{1}, i_{2}=1,2, \ldots, p$, define

$$
\begin{aligned}
U_{i}^{*}\left(\mathbf{\Psi}_{6}^{p}\right) & =\left[\left(\frac{\mathbf{R}}{n}\right)^{i-1}\right]^{\mathrm{t}}\left(\mathbf{O}-\mathbf{E}_{0}\right) ; \\
I_{i_{1} i_{2}}^{*}\left(\mathbf{\Psi}_{6}^{p}\right) & =\left[\left(\frac{\mathbf{R}}{n}\right)^{i_{1}-1}\right]^{\mathrm{t}} \mathbf{V}_{0}\left[\left(\frac{\mathbf{R}}{n}\right)^{i_{2}-1}\right] .
\end{aligned}
$$

- Test statistic induced by $\boldsymbol{\Psi}_{6}^{p}$ is

$$
S^{2}\left(\mathbf{\Psi}_{6}^{p}\right)=\left[\left(\mathbf{U}^{*}\left(\mathbf{\Psi}_{6}^{p}\right)\right)\right]^{\mathrm{t}}\left[\left(\mathbf{I}^{*}\left(\mathbf{\Psi}_{6}^{p}\right)\right)\right]^{-}\left[\left(\mathbf{U}^{*}\left(\mathbf{\Psi}_{6}^{p}\right)\right)\right] .
$$

## 7. Testing for Geometric Distribution

- Geometric is discrete analog of exponential distribution. As such it is a common model for discrete failure-time data.
- For $T \sim \operatorname{GEOM}(\eta)$, its pdf and df are

$$
\begin{aligned}
p(j \mid \eta) & =(1-\eta)^{j-1} \eta, j=1,2, \ldots \\
F(j \mid \eta) & =1-(1-\eta)^{j}, j=1,2, \ldots
\end{aligned}
$$

- Hazard rates are $\lambda(j \mid \eta)=\eta, j=1,2, \ldots$.
- Problem: To test $\operatorname{GEOM}\left(\eta_{0}\right)$ based on a right-censored data $\left(Z_{1}, \delta_{1}\right),\left(Z_{2}, \delta_{2}\right), \ldots,\left(Z_{n}, \delta_{n}\right)$ with

$$
Z_{i}=\min \left(T_{i}, C_{i} \wedge J\right), i=1,2, \ldots, n
$$

where $C_{1}, C_{2}, \ldots, C_{n}$ are censoring variables, and $J$ is the upper limit of the observation period.

- Specialized test statistics obtained from earlier expressions by taking $E_{j}^{0}=R_{j} \eta_{0}$ for $j=1, \ldots, J$.
- For instance, with smoothing order $p$,

$$
\begin{aligned}
& S^{2}\left(\mathbf{\Psi}_{6}^{p}\right)=\frac{n}{\eta_{0}\left(1-\eta_{0}\right)}\left[\left(\sum_{j=1}^{J}\left(\frac{R_{j}}{n}\right)^{i}\left(\hat{\lambda}_{j}-\eta_{0}\right)\right)_{i=1, \ldots, p}\right]^{\mathrm{t}} \times \\
& \quad\left[\left(\sum_{j=1}^{J}\left(\frac{R_{j}}{n}\right)^{i_{1}+i_{2}-1}\right)_{i_{1}, i_{2}=1, \ldots, p}\right]^{-}\left[\left(\sum_{j=1}^{J}\left(\frac{R_{j}}{n}\right)^{i}\left(\hat{\lambda}_{j}-\eta_{0}\right)\right)_{i=1, \ldots, p}\right]
\end{aligned}
$$

## 8. Simulation Studies

- Simulation studies under the $\operatorname{GEOM}\left(\eta_{0}\right)$ null hypothesis.
- Tests based on $S^{2}\left(\psi_{1}\right), S^{2}\left(\psi_{2}\right), S^{2}\left(\psi_{3}^{1}\right), S^{2}\left(\psi_{3}^{-1}\right), S^{2}\left(\boldsymbol{\Psi}_{5}\right)$, $S^{2}\left(\mathcal{I}_{J_{0}}\right)$, and $S^{2}\left(\boldsymbol{\Psi}_{6}^{p}\right)$ for $p=1,2,3,4$.
- For statistic $S^{2}\left(\mathbf{\Psi}_{5}\right)$ with a general partition $A_{1}, A_{2}, \ldots, A_{p}$, we chose the four partitions of $\mathcal{J}=\{1,2, \ldots, J\}$ :

1. for $S^{2}\left(\boldsymbol{\Psi}_{5}^{1}\right)$ :"odds" and "evens."
2. for $S^{2}\left(\boldsymbol{\Psi}_{5}^{2}\right)$ : divide into two parts.
3. for $S^{2}\left(\mathbf{\Psi}_{5}^{3}\right)$ : divide into three parts.
4. for $S^{2}\left(\boldsymbol{\Psi}_{5}^{4}\right)$ : divide into four parts.

## - Level Simulations:

- $T_{1}, T_{2}, \ldots, T_{n} \operatorname{IID} \operatorname{GEOM}\left(\eta_{0}\right)$.
- $C_{1}, C_{2}, \ldots, C_{n} \operatorname{IID} \operatorname{GEOM}(\eta)$ with $\eta$ chosen so $\mathbf{P}\left\{T_{1} \leq C_{1}\right\}$ equals a specified value of UCP.
- $J$ fixed, so effective probability of uncensored observation is $\mathbf{P}\left\{T_{1} \leq C_{1} \wedge J\right\}$.
- Parameters Values: Combinations of $n \in\{30,50,100,200\}$, $J \in\{30,50\}, \eta_{0} \in\{.03, .10\}, \mathrm{UCP} \in\{.50, .75\}$, and with $\mathrm{MReps}=1000$.
- Typical result of level runs: $\eta_{0}=.03$ and UCP $=75 \%$.

| $J_{0}$ | 30 |  |  |  | 50 |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | 30 | 50 | 100 | 200 | 30 | 50 | 100 | 200 |
| \%UCP | 52.8 | 53.1 | 52.9 | 52.9 | 65.1 | 65.3 | 65.5 | 65.3 |
| Statistic |  |  |  |  |  |  |  |  |
| $S^{2}\left(\psi_{1}\right)$ | 5.8 | 4.8 | 5.5 | 5.5 | 5.4 | 5.3 | 4.9 | 5.8 |
| $S^{2}\left(\psi_{2}\right)$ | 5.2 | 4.7 | 6.1 | 5.6 | 4.9 | 5.1 | 4.8 | 5.8 |
| $S^{2}\left(\psi_{3}^{1}\right)$ | 5.7 | 5.3 | 4.5 | 4.5 | 6.0 | 5.4 | 5.9 | 5.4 |
| $S^{2}\left(\psi_{3}^{-1}\right)$ | 4.7 | 5.0 | 5.9 | 6.1 | 3.3 | 4.0 | 4.8 | 5.3 |
| $S^{2}\left(\Psi_{5}^{1}\right)$ | 5.4 | 5.0 | 4.9 | 6.1 | 5.9 | 6.1 | 5.5 | 5.9 |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{2}\right)$ | 5.5 | 4.9 | 5.1 | 5.1 | 5.3 | 5.9 | 4.6 | 5.5 |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{3}\right)$ | 7.7 | 5.2 | 5.0 | 5.0 | 7.2 | 6.9 | 5.5 | 6.1 |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{4}\right)$ | 6.8 | 5.3 | 6.0 | 5.2 | 7.3 | 6.2 | 5.8 | 4.6 |
| $S^{2}\left(\mathcal{I}_{J_{0}}\right)$ | 11.0 | 8.9 | 7.5 | 5.6 | 13.3 | 10.2 | 8.5 | 6.3 |
| $S^{2}\left(\boldsymbol{\Psi}_{6}^{1}\right)$ | 5.8 | 4.8 | 5.5 | 5.5 | 5.4 | 5.3 | 4.9 | 5.8 |
| $S^{2}\left(\Psi_{6}^{2}\right)$ | 6.3 | 6.2 | 5.3 | 5.1 | 5.8 | 5.9 | 5.6 | 6.0 |
| $S^{2}\left(\boldsymbol{\Psi}_{6}^{3}\right)$ | 7.0 | 5.4 | 5.7 | 5.1 | 8.0 | 6.3 | 6.9 | 5.8 |
| $S^{2}\left(\boldsymbol{\Psi}_{6}^{4}\right)$ | 8.2 | 6.4 | 5.9 | 4.6 | 8.2 | 6.7 | 6.2 | 5.2 |

- Test based on $S^{2}\left(\mathcal{I}_{J}\right)$ did not achieve specified level. Too anticonservative. Could be due to the asymptotic approximation being poor.
- All other tests seem to achieve the specified level, especially when $n \geq 100$.


## - Power Simulations:

- Same geometric null hypothesis.
- Families of alternatives considered:

1. Geometric (of course, with different mean than the null);
2. Poisson;
3. Negative Binomial;
4. Polynomially-generated hazards of form

$$
\begin{gathered}
\lambda_{j}=2 \eta_{0} \frac{G_{j}(\mathbf{a})}{\left[1+G_{j}(\mathbf{a})\right]}, j=1,2, \ldots, J, \\
G_{j}(\mathbf{a})=\exp \left\{\sum_{k=1}^{q} a_{k}\left(\frac{j}{J}\right)^{k-1}\right\} .
\end{gathered}
$$

5. Trigonometrically-specified hazards of form

$$
\begin{gathered}
\lambda_{j}=2 \eta_{0} \frac{G_{j}(\mathbf{a}, \mathbf{b}, \mathbf{c})}{\left[1+G_{j}(\mathbf{a}, \mathbf{b}, \mathbf{c})\right]}, j=1,2, \ldots, J, \\
G_{j}(\mathbf{a}, \mathbf{b}, \mathbf{c})=\exp \left\{\sum _ { k = 1 } ^ { q } \left[a_{k} \sin \left\{2 \pi c_{k}\left(\frac{j}{J}\right)\right\}+\right.\right. \\
\left.\left.b_{k} \cos \left\{2 \pi c_{k}\left(\frac{j}{J}\right)\right\}\right]\right\} .
\end{gathered}
$$

- Instead of presenting detailed results of achieved powers, we'll just present the tests' "final grades" (as most of us are good at assigning grades)!


## - 'Final Grades' of Tests:

- Letter grades of the tests under five classes of alternatives based on simulated powers. $\mathrm{A}=\mathrm{Best}, \mathrm{E}=$ Worst.

| Type of <br> Alt. | Geom. | Neg. <br> Bin. | Poisson | Poly. <br> Gen. | Trig. <br> Gen. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Statistic |  |  |  |  |  |
| $S^{2}\left(\psi_{1}\right)$ | A | E | E | E | E |
| $S^{2}\left(\psi_{2}\right)$ | $\mathrm{A}-$ | E | E | E | E |
| $S^{2}\left(\psi_{3}^{1}\right)$ | $\mathrm{A}-$ | E | B | D | D |
| $S^{2}\left(\psi_{3}^{-1}\right)$ | B | E | $\mathrm{A}-$ | E | E |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{1}\right)$ | B | E | E | E | E |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{2}\right)$ | B | B | E | C | E |
| $S^{2}\left(\mathbf{\Psi}_{5}^{3}\right)$ | B | B | D | B | D |
| $S^{2}\left(\boldsymbol{\Psi}_{5}^{4}\right)$ | $\mathrm{B}-$ | B | B | B | B |
| $S^{2}\left(\boldsymbol{\Psi}_{6}^{1}\right)$ | A | E | E | E | E |
| $S^{2}\left(\boldsymbol{\Psi}_{6}^{2}\right)$ | B | A | A | B | C |
| $S^{2}\left(\mathbf{\Psi}_{6}^{3}\right)$ | B | A | A | A | $\mathrm{B}-$ |
| $S^{2}\left(\mathbf{\Psi}_{6}^{4}\right)$ | $\mathrm{B}-$ | $\mathrm{A}-$ | A | A | B |

- Some conclusions from simulation results.
- Unless there is specific knowledge of the type of alternative, prudent to utilize $S^{2}\left(\boldsymbol{\Psi}_{6}^{p}\right)$ for $p=2,3,4$, with higher values of $p$ preferred when hazard sequence is expected to have high frequency.
- Tests based on uni-dimensional smoothing functions ( $p=1$ ) have good powers against specific types of departures from null hypothesis, but poor powers for other types of alternatives.
- Those based on $S^{2}\left(\boldsymbol{\Psi}_{6}^{p}\right)$ with $p=2,3,4$, though not always achieving highest powers, have competitive powers for all the alternative classes, hence could serve as omnibus tests.
- Though Pearson-type tests have decent powers, still are bettered by those based on the $\boldsymbol{\Psi}_{6}$ (at-risk based polynomial) specification.


## 9. Concluding Remarks

- Performance under different null hypothesis?
- Composite null case. Issue of plug-in procedure. Forthcoming.
- (Locally) optimal smoothing matrix for given null and alternative? Local powers?
- An automated or data-driven method for deciding on $p$ ?
- If asymptotic approximations not satisfactory, computer-intensive approaches?
- Comparisons with future generalizations of gof tests for censored and continuous data?

