Estimating Load-Sharing Properties in a Dynamic Reliability System

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Modeling Dependence Between Components

Most reliability methods are intended for components that operate independently within a system.

It is more realistic, however, to develop models that incorporate stochastic dependencies among the system's components. Options for modeling dependent systems:

- Shock models.
- Load-share models.

Load Sharing Models

- Load share models dictate that component failure rates depend on the operating status of the other system components and the effective system structure function.

- Daniels (1945) originally adopted this model to describe how the strain on yarn fibers increases as individual fibers within a bundle break.

- A bundle of fibers can be considered as a parallel system subject to a steady tensile load.

The Load-Share Rule

The most important element of the load-share model is the rule that governs how failure rates change after some components in the system fail.

- **Equal Load Share Rule**: A constant system load distributed equally among the working components
- **Local load sharing rule**: A failed component's load is transferred to adjacent components.
- **Monotone load sharing rule**: The load on any individual component is nondecreasing as other items fail.

Past research has stressed reliability estimation based on *known* load share rules

Examples of Load-Share Systems

- **Textile Engineering**:
- **Nuclear Reactor Safety**: Failure of one back-up system adversely affects another
- **Software Reliability**: Discovery of a major software defect can help reveal or conceal other existing bugs
- **Civil Engineering**: Welded joints on large support structures
- **Materials Testing**: Fatigue and crack growth
- **Population Sampling**: Capture/Recapture methods
- **Combat Modeling**: Loss of component in combat affects death rate of others

An Unknown Load-Share Rule

- Past research emphasizes load-share modeling based on known load share rules.
- In these examples, the load-share rule might be unknown.
- Our focus: Case in which the system is governed by an unknown equal load-share rule.
- General set up: Observe component lifetimes in parallel systems of identical components .

Estimation of Load-Share Model Parameters

- Observe n i.i.d. systems of k components.
- For $i = 1, 2, 3, \ldots$, let $S_{i,1} < S_{i,2} < \ldots$ be the successive component failure times for the ith system
- F represents the baseline component failure time distribution function.
- Hazard function corresponding to F is $R(x) = -\log(1 - F(x))$
- Hazard rate is $r(x) = f(x)/[1 - F(x)]$, where $f(x)$ is the density of F.

Load Share Parameters

Until the first component failure, the failure rate of each of k components in the system equals the baseline rate $r(x)$.

Upon the first failure within a system, the failure rates of the $k-1$ remaining components jump to $\gamma_1 r(x)$, and remain at that rate until the next component failure.

After this failure, the failure rates of the $k-2$ surviving components jump to $\gamma_2 r(x)$, and so on.

Load Share Parameters

The (equal) load share rule can be characterized by the $k-1$ unknown parameters $\boldsymbol{\gamma}$ = $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ and the unknown baseline distribution or hazard function.

For example, a system with a constant load would assign

$$\gamma_j = k/(k-j), \quad j = 1, ..., k-1$$

Maximum Likelihood Estimation of R and γ

In the ith system, the conditional hazard function of the $(j+1)$ smallest component lifetime $S_{i,j+1}$, given the first i component failure times $S_{i,1},\ldots,S_{i,j}$, is (for $s > S_{i,j}$)

$$R^*(s|S_{i,1},\ldots,S_{i,j}) =$$

$$\gamma_j R(s) + (\gamma_{j-1} - \gamma_j)R(S_{i,j}) + \ldots + (1-\gamma_1)R(S_{i,1})$$

The corresponding likelihood function, in terms of R^*, is

$$\prod_{i=1}^{n}\prod_{j=1}^{k} dR^*(S_{ij})\exp\{-R^*(S_{ij})\}.$$

Computing the MLE

Standard approach for finding the MLE:

- Fix $\boldsymbol{\gamma}$.
- Maximize likelihood with respect to R to obtain $\hat{R}(\cdot;\boldsymbol{\gamma})$
- Plug $\hat{R}(\cdot;\boldsymbol{\gamma})$ in to obtain the profile likelihood for $\boldsymbol{\gamma}$
- Compute the MLE $\hat{\boldsymbol{\gamma}}$; final estimator of $R(\cdot)$ is $\hat{R}(\cdot;\hat{\boldsymbol{\gamma}})$

To understand properties of the nonparametric MLE, we model the load-share system using counting processes.

Notation for Counting Processes

- $N_i(t) = \sum_{j=1}^{k} I(S_{i,j} \leq t), \quad i = 1, 2, \ldots, n$
- $\gamma[N_i(w)] = \sum_{j=0}^{k-1} \gamma_j I(N_i(w) = j)$
- $Y_i(w) = (k - N_i(w-))\, I(\tau \geq w)$
- $A_i(t) = \int_0^t \gamma[N_i(u-)] r(u) Y_i(u) du$

If γ is known, then analogous to the derivation of the the Nelson-Aalen estimator (with $J(w) = I(\sum_{i=1}^{k} Y_i(w) > 0)$), we obtain the estimator

$$\hat{R}(s; \boldsymbol{\gamma}) = \int_0^s \frac{J(w) dN(w)}{\sum_{i=1}^{n} Y_i(w) \gamma[N_i(w-)]}$$

MLE using Counting Processes

To obtain the estimator of $R(\cdot)$ for the more general case where γ is unknown, we first obtain the profile likelihood for γ by plugging in $\hat{R}(\cdot;\boldsymbol{\gamma})$ into the likelihood function

$$L_p(s;\boldsymbol{\gamma}) = \prod_{i=1}^{n} \prod_{0 \le w \le s} \left[\frac{Y_i(w)\gamma[N_i(w-)]}{\sum_{l=1}^{n} Y_l(w)\gamma[N_l(w-)]} \right]^{dN_i(w)}.$$

Once $\hat{\gamma}$ is obtained, the estimator of R becomes

$$\hat{R}(s) = \hat{R}(s;\hat{\gamma}).$$

MLE using Counting Processes

By virtue of the product representation of $\bar{F} = 1 - F$ in terms of R given by $\bar{F}(s) = \prod_{0 \le w \le s}[1 - R(dw)]$, we then obtain an estimator of $\bar{F}$ via

$$\hat{\bar{F}}(s) = \prod_{0 \le w \le s} \left[1 - \hat{R}(dw)\right].$$

Solving the MLE

The MLE can be computed by solving the set of k nonlinear equations

$$\boldsymbol{U}(\tau;\boldsymbol{\gamma}) = \sum_{i=1}^{n} \int_0^{\tau} \left[\frac{\boldsymbol{Q}_i(w)}{\boldsymbol{\gamma}'\boldsymbol{Q}_i(w)} - \frac{\boldsymbol{Q}(w)}{\boldsymbol{\gamma}'\boldsymbol{Q}(w)} \right] dN_i(w) = \mathbf{0}$$

where

- $Q_{i,j}(t) = Y_i(t)I(N_i(t-) = j)$, $1 \leq i \leq n, 0 \leq j \leq k-1$;
- $\boldsymbol{Q}_i(t) = (Q_{i,0}(t), ..., Q_{i,k-1}(t))'$, $1 \leq i \leq n$;
- $\boldsymbol{Q}(t) = (\sum_{i=1}^{n} Q_{i,0}(t), ..., \sum_{i=1}^{n} Q_{i,k-1}(t))'$;

(Solve with an iterative scheme; e.g. Newton-Raphson.)

Asymptotic Properties

Suppose we have that

- $\hat{\boldsymbol{\rho}}(t;\boldsymbol{\gamma}) = \sum_{i=1}^{n} \boldsymbol{\gamma} * \boldsymbol{Q}_i(t)/\boldsymbol{\gamma}'\boldsymbol{Q}(t)$;
- $\boldsymbol{\delta}_i(t) = (\delta_{i,0}(t), ..., \delta_{i,k-1}(t))'$, with $\delta_{i,j}(t) = I(Q_{i,j}(t) > 0), 1 \leq i \leq n$;
- $\boldsymbol{\Upsilon}(s;\boldsymbol{\gamma}) \equiv \int_0^s \left[\boldsymbol{D}(\boldsymbol{\rho}(w;\boldsymbol{\gamma})) - \boldsymbol{\rho}(w;\boldsymbol{\gamma})\boldsymbol{\rho}(w;\boldsymbol{\gamma})'\right] \boldsymbol{\gamma}'\boldsymbol{q}(w) dR(w)$.

Lemma:

If $\{N_i(\cdot), i = 1, ..., n\}$ are independent and identically distributed, and

$$\inf_{0 \le w \le \tau} \sum_{j=0}^{k-1} (k-j)\gamma_j P(N_1(w-) = j) > 0,$$

then $\boldsymbol{U}(s;\boldsymbol{\gamma}) = \sum_{i=1}^{n} \int_0^s [\boldsymbol{\delta}_i(w) - \hat{\boldsymbol{\rho}}(w;\boldsymbol{\gamma})] (dN_i(w) - dA_i(w))$ is a square-integrable martingale with quadratic variation process $\langle \boldsymbol{U}(\cdot;\boldsymbol{\gamma}) \rangle (s)$ whose in-probability limit is $\boldsymbol{\Upsilon}(s;\boldsymbol{\gamma})$. Furthermore, $n^{-1/2}\boldsymbol{U}(\cdot;\boldsymbol{\gamma})$ converges weakly to a zero-mean Gaussian process with covariance matrix function $\boldsymbol{\Upsilon}(\cdot;\boldsymbol{\gamma})$.

Theorem 1

Under the conditions of the Lemma,

(i) $\hat{\gamma}$ converges in probability to γ; and

(ii) $\sqrt{n}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}(\tau, \boldsymbol{\gamma}))$ where $\boldsymbol{\Sigma}(\tau, \boldsymbol{\gamma}) = \boldsymbol{D}(\boldsymbol{\gamma})\boldsymbol{\Upsilon}(\tau, \boldsymbol{\gamma})^{-1}\boldsymbol{D}(\boldsymbol{\gamma})$, and with

$$\boldsymbol{\Upsilon}(\tau, \boldsymbol{\gamma}) \equiv \int_0^\tau \left[\boldsymbol{D}(\boldsymbol{\rho}(w; \boldsymbol{\gamma})) - \boldsymbol{\rho}(w; \boldsymbol{\gamma})\boldsymbol{\rho}(w; \boldsymbol{\gamma})'\right] \boldsymbol{\gamma}'\boldsymbol{q}(w) dR(w).$$

where

- $\boldsymbol{\rho}(t; \boldsymbol{\gamma}) = E[\sum_{i=1}^n \boldsymbol{\gamma} * \boldsymbol{Q}_i(t)]/E[\boldsymbol{\gamma}'\boldsymbol{Q}(t)]$ = $\boldsymbol{\gamma} * \boldsymbol{q}(t)(\boldsymbol{\gamma}'\boldsymbol{q}(t))^{-1}$ and
- $\boldsymbol{q}(s) = (q_0(s), \ldots, q_{k-1}(s))$, with $q_j(w) = E(Q_{i,j}(w))$ = $(k-j)P(\tau \geq w, N_1(w-) = j)$.

Theorem 2

Under the conditions of the Lemma, if τ is such that $\boldsymbol{\gamma}'\boldsymbol{q}(\tau) > 0$, then

$$\left\{\sqrt{n}(\hat{R}(s) - R(s)) : 0 \leq s \leq \tau\right\}$$

converges weakly to a zero-mean Gaussian process with variance function

$$\Xi(s;\boldsymbol{\gamma}) \equiv \int_0^s \{\boldsymbol{\gamma}'\boldsymbol{q}(w)\}^{-1} dR(w) + \boldsymbol{\varrho}(s;\boldsymbol{\gamma})'[\boldsymbol{\Upsilon}(\tau;\boldsymbol{\gamma})]^{-1}\boldsymbol{\varrho}(s;\boldsymbol{\gamma}),$$

where $\boldsymbol{\varrho}(s;\boldsymbol{\gamma}) = \int_0^s \boldsymbol{\rho}(w;\boldsymbol{\gamma}) dR(w)$.

Corollary

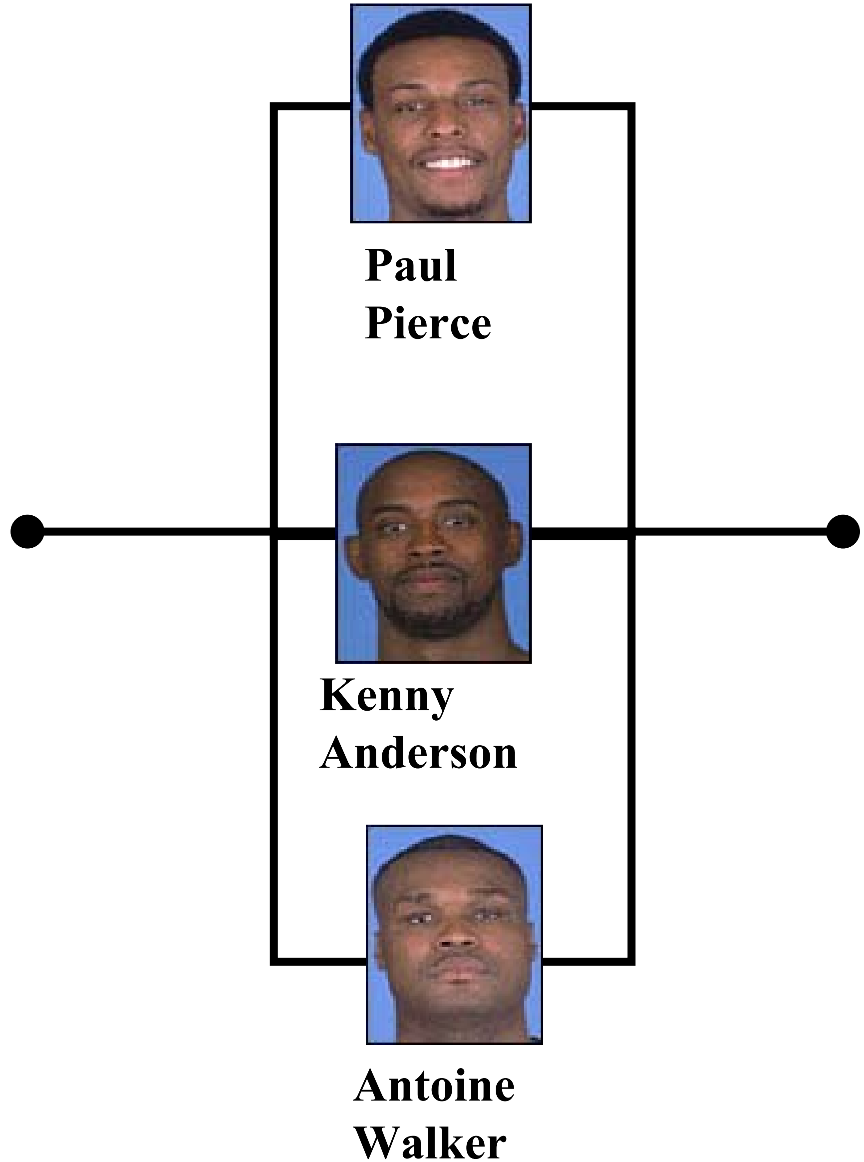
Under the conditions of Theorem 2,

$$\left\{\sqrt{n}(\hat{\bar{F}}(s) - \bar{F}(s)) : 0 \leq s \leq \tau\right\}$$

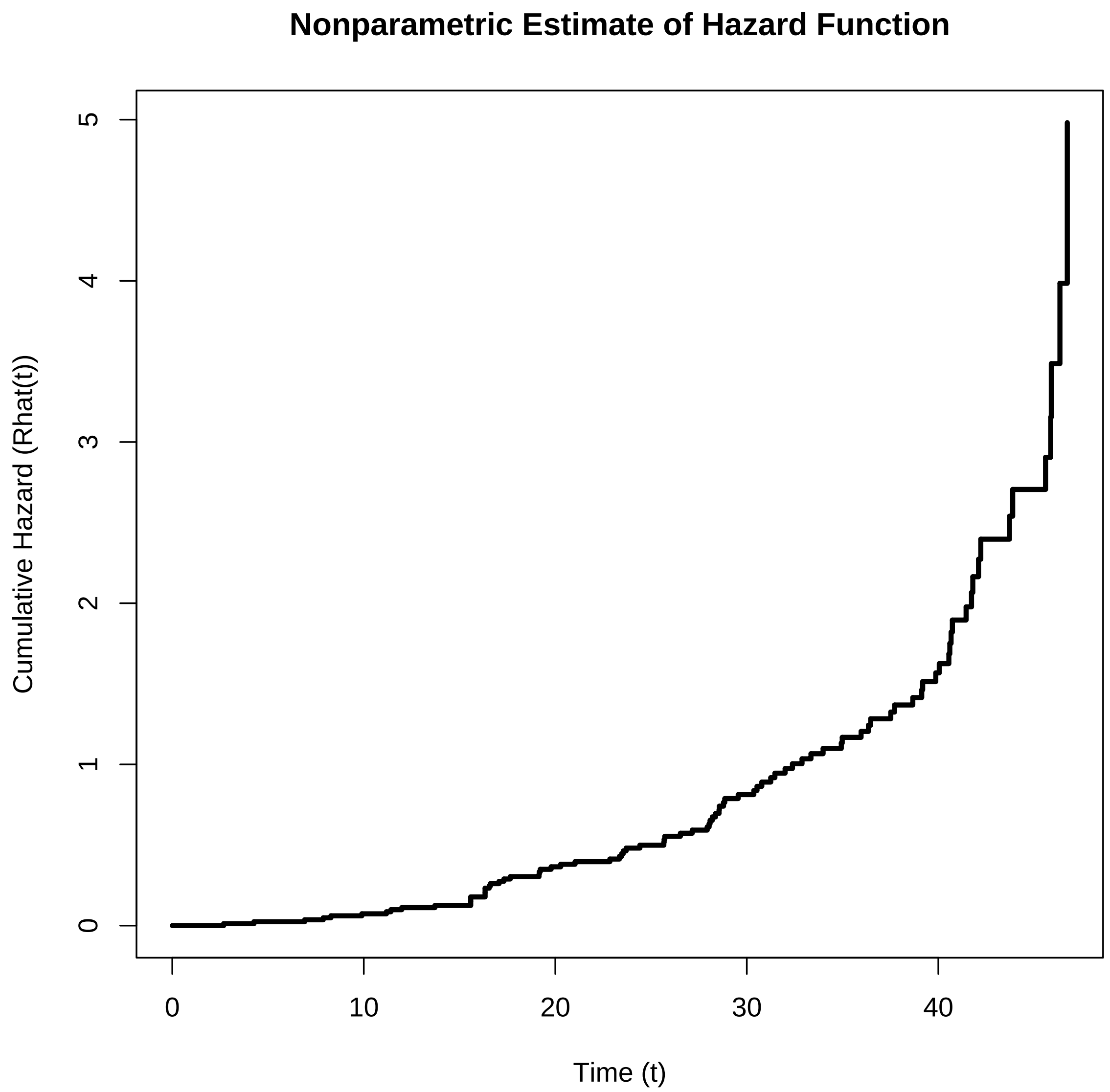
converges weakly to a zero-mean Gaussian process $\{Z(s) : 0 \leq s \leq \tau\}$ whose variance function is var$\{Z(s)\} = \bar{F}(s)^2 \, \Xi(s; \boldsymbol{\gamma})$.

3-Component Parallel System:

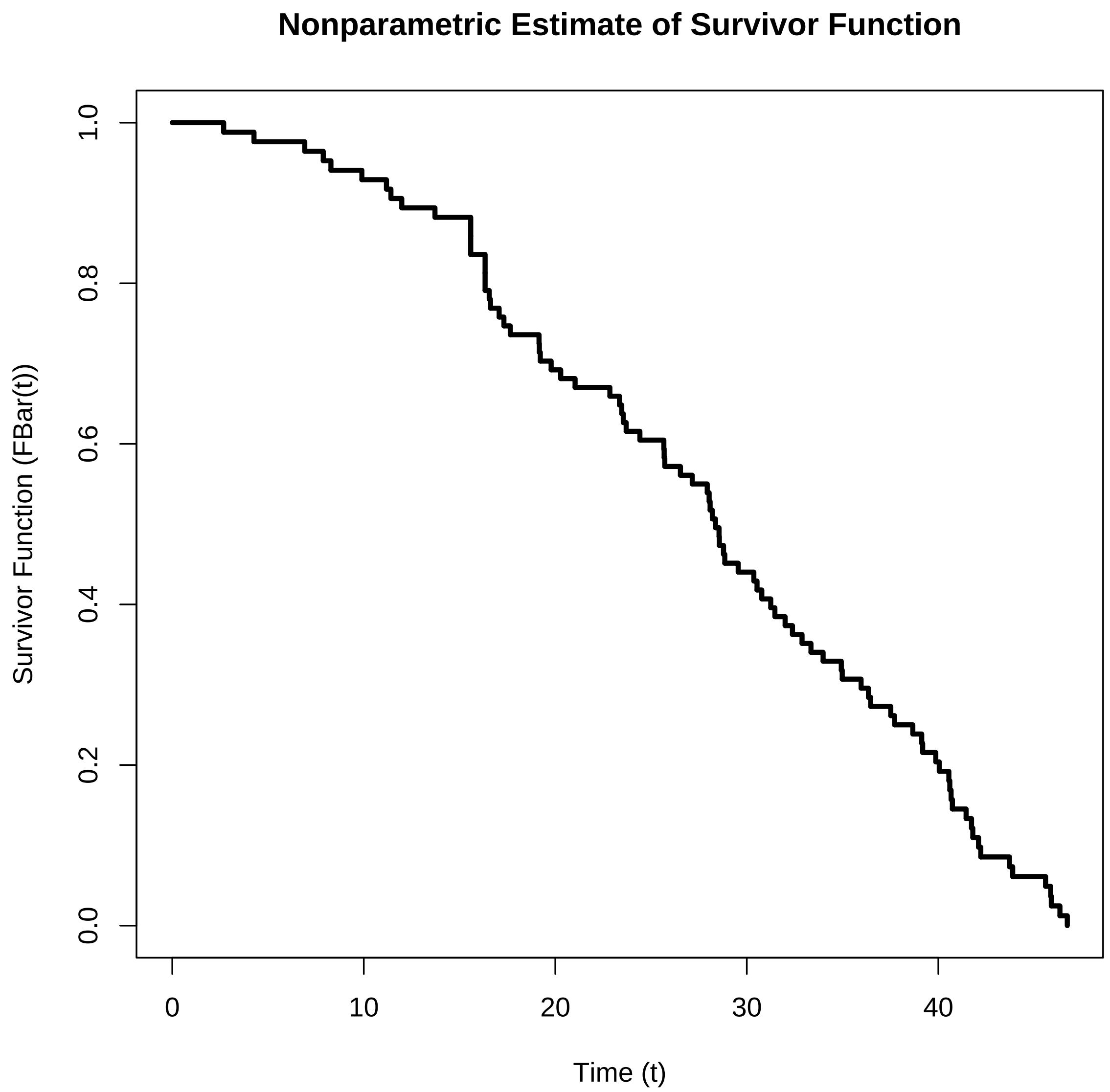
2000-2001 Boston Celtics



Estimated Cumulative Hazard: Minutes Played Until 2^{nd} Foul



Estimated Survivor Function of Minutes Played



Confidence Regions (50%, 90%, 95%) for (γ_1, γ_2)