Accelerated Testing with Recurrent Events

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2007 IBS Eastern North American Region Meeting
Hyatt Regency @ Atlanta, GA
Research support provided by NIH Grant 2 R01 GM56182
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March 13, 2007

Outline

- Introduction
- Model
- Methods
- Optimal Design
- Future Research

Accelerated Life Testing

- Accelerated Life Testing (ALT) induces failures through high stress (temperature, humidity, usage etc.)
- Stresses are joined through a link function (inverse power, log-linear, Arrhenius, etc.)
- Results used designing warranties, system comparison, and failure detection

Example Accelerated Life Testing

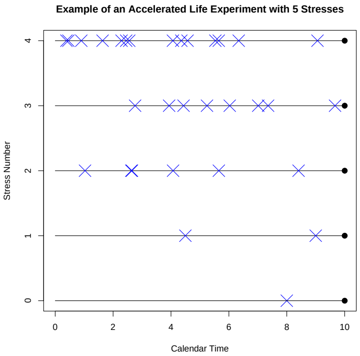

Figure: Graphical view of an ALF with 5 Stresses with one site for each stress.

Previous Literature

- Chernoff ('62) - simultaneous testing with type I censoring and successive testing with complete data under HPP
- Nelson and Kielpinski ('76); Nelson and Meeker ('78)- type I censored simultaneous testing for Normal, Log-Normal, Weibull, and extreme value distributions

Problem Statement

Goals:

1. Parameter Estimation
2. Hypothesis tests and CI estimation on the percentile of the assumed life distribution at the operating stress level
3. Develop optimum ALT test plans

Notation

1. i = stress, j = site, l = event.
2. Counting Process

$$N^{\dagger}_{ij}(s) = \sum_{l=1}^{\infty} I(S_{ijl} \leq s, S_{ijl} \leq \tau_i)$$

3. At-risk indicator

$$Y^{\dagger}_{ij}(s) = I\{\tau_i \geq s\}$$

4. Backward recurrence time

$$\varepsilon_{ij}(w) = w - S_{iN^{\dagger}_{ij}(w-)}$$

Basic Assumptions

- Generally the distribution we are considering is:

$$T_{ijl}|_{x_i} \overset{iid}{\longrightarrow} \Lambda_0\left[\frac{t}{\theta(x_i;\beta)};\alpha\right] \quad where \quad log[\theta(x_i;\beta)] = \beta_0 + \beta_1 x_i$$

- Assume $X_0, X_1, X_2, \ldots, X_p$, possible stresses:
  - $X_p \equiv X_H$ denotes the highest stress
  - X_0 is the design stress level
- For the i^{th} stress we have m_i systems with:

$$m_0, m_1, m_2, \ldots, m_p \quad M = \sum_{i=0}^{p} m_i$$

AFT Model

- The compensator for the i^{th} stress and j^{th} site will be:

$$A_{ij}^{\dagger}(s;\alpha,\beta_0,\beta_1) = \int_0^s Y_{ij}^{\dagger}(v)\frac{1}{\theta(x_i;\beta_0,\beta_1)}\lambda_0\left[\frac{\varepsilon_{ij}(v)}{\theta(x_i;\beta_0,\beta_1)};\alpha\right]dv$$

which is predictable with respect to the common filtration:

$$\mathcal{F}_s = \mathcal{F}_0 \vee \left\{\bigvee_{i=1}^{n} \sigma\left\{\left(N_{ij}^{\dagger}(w), Y_{ij}^{\dagger}(w+), \varepsilon_{ij}(w+)\right) : 0 < w \leq s\right\}\right\}$$

- $M_{ij}^{\dagger}(s) = N_{ij}^{\dagger}(s) - A_{ij}^{\dagger}(s)$ is a zero-mean square integrable martingale for the i^{th} stress and j^{th} site.

Maximum Likelihood Formulation

- Following Jacod ('75):

$$L_{ij}(s; x_i, \beta, \alpha) = \prod_{i=1}^{n} \left\{ \left[d\, A_{ij}^{\dagger}(s; x_i, \beta, \alpha) \right]^{\Delta N_{ij}^{\dagger}(s)} \exp\left[-A_{ij}^{\dagger}(s; x_i, \beta, \alpha) \right] \right\}$$

- Score Process:

$$U(s; x_i, \beta, \alpha) = \begin{bmatrix} U_{\beta}(s; x_i, \beta, \alpha) \\ U_{\alpha}(s; x_i, \alpha, \beta) \end{bmatrix} = \sum_{i=1}^{p} \sum_{j=1}^{m_i} \int_0^s \mathbf{H}\left[\frac{\varepsilon_{ij}(w)}{\theta(x_i; \beta)}; x_i, \beta, \alpha \right] dM_{ij}^{\dagger}(w; x_i, \beta, \alpha)$$

MLE Result:

- Subject to Borgan('84):

$$\sqrt{n}[\hat{\theta} - \theta] \xrightarrow{iid} AN(0, \Sigma^{-1})$$

  where $\hat{\theta} = (\hat{\alpha} \quad \hat{\beta}_0 \quad \hat{\beta}_1)'$ is the Maximum Likelihood estimator of $\theta = (\alpha \quad \beta_0 \quad \beta_1)'$.

- A consistent estimate of Σ is:

$$\hat{\Sigma} = \frac{1}{M}\langle \mathbf{U}(s; \beta, \alpha)\rangle = \frac{1}{M}\sum_{i=1}^{p}\sum_{j=1}^{m_i}\int_0^s \mathbf{H}_{ij}\left[\frac{\varepsilon_{ij}(w)}{\theta(x_i;\beta)}; x_i, \beta, \alpha\right]^{\otimes 2} dA_{ij}^{\dagger}(w; x_i, \beta, \alpha)$$

MLE result: H matrix

- Using the notation that;

$$\begin{aligned}\gamma(x_i,\beta) &= \nabla_\beta Log[\theta(x_i;\beta)]\\ \pi(t,\alpha) &= \nabla_\alpha Log[\lambda_0(t;\alpha)]\\ \eta(t,\alpha) &= \frac{\partial}{\partial t} Log[\lambda_0(t;\alpha)]\end{aligned}$$

We have:

$$\mathbf{H}_{ij}\left[\frac{\varepsilon_{ij}(w)}{\theta(x_i;\beta)};x_i,\beta,\alpha\right] = \begin{pmatrix} -\gamma(x_i;\beta)\left[1+\frac{\varepsilon_{ij}(w)}{\theta(x_i;\beta)}\eta(\frac{\varepsilon_{ij}(w)}{\theta(x_i;\beta)},\alpha)\right] \\ \pi\left[\frac{\varepsilon_{ij}(w)}{\theta(x_i;\beta)},\alpha\right] \end{pmatrix}$$

Maximum Likelihood Formulation

- After calculations:

$$\langle \mathbf{U}_{ij}(s;\beta,\alpha)\rangle = \int_0^{\tau} Y_{ij}(w)\mathbf{H}_{ij}(w;x_i,\beta,\alpha)^{\otimes 2}\lambda_0(w;\alpha)dw$$

Important Regularity Condition

- Covariation Process:

$$\frac{1}{M}\langle \mathbf{U}(s;\beta,\alpha)\rangle = \frac{1}{M}\sum_{i=1}^{p}\sum_{j=1}^{m_i}\int_0^{\tau} Y_{ij}(w)\mathbf{H}_{ij}(w;x_i,\beta,\alpha)^{\otimes 2}\lambda_0(w;\alpha)dw$$

$$as \quad M \to \infty \quad \xrightarrow{d} \sum_{i=1}^{p} S_i \int_0^{\tau} y(w;\beta,x_i)G(w;x_i,\beta,\alpha)\lambda_0(w;\alpha)dw$$

$$= \sum_{i=1}^{p} Q(x_i;\beta,\alpha)$$

Where $S_i \equiv \frac{m_i}{M}$:

$$E[Y_{ij}(s,t)] = y(w;\beta,x_i) = \bar{F}[t\theta(x_i)] + \bar{F}[t\theta(x_i)]\rho[\theta(x_i)(\tau - t)]$$

[Pena et al. 2001, JASA]

Optimization Criteria

- Most often the goal in ALT is to estimate a percentile (q) of the CDF at the operating stress level:

$$\xi_q(X_0) \;=\; F_0^{-1}(q;X_0;\beta_0,\beta_1,\alpha) \tag{1}$$

- Minimization Criteria:

$$Min_{\{S_i,i=1,2,\ldots,p\}}\left\{Var\left[\xi_q(X_0)\right]\right\}$$

An application of the Delta Method yields:

$$Var(\xi_q(X_0)) = B\Sigma^{-1}(s)B^t \quad with \quad B = \begin{pmatrix} \nabla_\beta \xi_q(X_0) \\ \nabla_\alpha \xi_q(X_0) \end{pmatrix}$$

Previous Results

- Elfving's ('52) result was then used by Chernoff ('62) to develop the first ALT 'locally optimal' test plans
- Chernoff found that in a two-parameter life-stress relationship only two stresses need to be used, with X_H being one

Compromised Plans

- Statistically optimal test plans will always use a minimum amount of stresses, but can are not robust to model misspecifications
- In a two-parameter model Compromised Plans use three or four test stresses

Future Research

- Develop statistically optimum and compromised plans for exponential, and weibull distributions under a recurrent event framework
- Develop hypothesis tests and confidence intervals for $\xi_p(\cdot)$
- Develop a nonparametric model under a recurrent event framework

Thank You

Thanks For Listening