Chapter 6: Continuous-Time Markov Chains

- In Chapter 4, we studied Markov chains $\{X_n\}$ with a discrete index set $n = 0, 1, 2, \ldots$
- We now study continuous-time Markov chains $\{X(t), t \geq 0\}$ in which the index set is all non-negative real numbers.
- These processes have the Markovian property: Their future behavior depends only on the present, not the past.

6.2 Continuous-Time Markov Chains

Defn: A stochastic process $\{X(t), t \geq 0\}$ whose state space is the set of non-negative integers, is a continuous-time Markov chain if for all $s,t \ge 0$ and all $i, j, x(u), 0 \le u \le s$:

- Hence the conditional distribution of the process at time $t+s$ depends only on the state at time s, for all $s,t \ge 0$.
- If $P[X(t+s)=j \mid X(s)=i]$ does not depend on s, then the chain has

_______________ transition probabilities.

Note: If the chain enters state i, let T_i denote the amount of time it spends in i before transitioning to a new state.
- By the Markovian property, future behavior only depends on where the chain is currently.

Therefore:

- This implies that T_i is ____________ and so it must be distributed as

________________.

- Hence we can alternately define a continuous-time Markov chain as a process which, when it enters some state i:

(1)

and (2)

- Note that which state j is visited after state i <u>cannot</u> depend on the amount of time spent in state i (otherwise, this would violate the Markovian property).

Example: A drive-through restaurant has two windows, one for ordering food and one for paying and picking up the food. Suppose the service times (in minutes) at windows 1 and 2 are independent exponential r.v.'s, with rates 1 and $\frac{1}{3}$, respectively. Suppose customers arrive at the restaurant according to a Poisson process with rate $\lambda = 0.5$ per minute, and that customers will leave immediately if either window is occupied by another customer. Model this with a continuous-time Markov chain.

6.3 Birth and Death Processes

- Consider a system in which the state at time t is the number of people n in the system at time t.
- Suppose further:
  (i) people enter the system with interarrival times that are exponential with rate λ_n
  (ii) people depart the system with interdeparture times that are exponential with rate μ_n
  (iii) arrivals and departures are independent.

- This process $\{X(t)\}$ is a <u>birth and death process</u> with arrival (or birth) rates $\{\lambda_n\}$ and departure (or death) rates $\{\mu_n\}$.
- The state space of $\{X(t)\}$ is:
- What are the $\{v_j\}$ and $\{P_{ij}\}$ of this process?

- Note that a *Poisson process* is simply a birth and death process with

- A *pure birth* process is a birth and death process with
- So the Poisson process is

- Another pure birth process is the *Yule process*: It has

- The Yule process arises if each of n individuals has exponential birth rate λ, making the birth rate of the total population ________ .