Thus the change in price

10.8 Stationary Processes

Defn. A stochastic process $\{X(t), t \geq 0\}$ is called a stationary process if for any n and $t_1, \dots, t_n$, the random vectors $(X(t_1), \dots, X(t_n))$ and $(X(t_1+s), \dots, X(t_n+s))$ have the same joint distribution for all s.

- That is, the joint distribution is the same, no matter the "starting point."
- Consider two examples of stationary processes that we have already seen:

Example 1: An ergodic continuous-time Markov chain $\{X(t)\}$ where $P[X(0)=j] = P_j$, $j \geq 0$, where $\{P_j\}$ are the limiting probabilities.

Example 2: $\{X(t)\}$, where $X(t) = N(t+L) - N(t)$, $t \geq 0$, where $L > 0$ is a constant and $\{N(t)\}$ is a Poisson process with rate λ.

- In Example 2, $X(t)$ counts the number of events in

Example (Random Telegraph Signal Process):

- Let $\{N(t)\}$ be a Poisson process with rate λ. Let X_0 be independent of $\{N(t)\}$ and have probability distribution $P[X_0=1]=P[X_0=-1]=\frac{1}{2}$. If $X(t) = X_0(-1)^{N(t)}$, then $\{X(t), t \geq 0\}$ is a random telegraph signal process.

- At any time t, $X(t)$ is equally likely to
- Since a Poisson process is stationary, $\{X(t)\}$ is also a stationary process.

$E[X(t)] =$

and $\mathrm{cov}[X(t), X(t+s)] =$

Defn. A stochastic process $\{X(t)\}$ is **weakly stationary** if $E[X(t)] = c$ and $\mathrm{cov}[X(t), X(t+s)]$ does not depend on t.

- That is, if $E[X(t)]$ and $\mathrm{var}[X(t)]$ do not depend on t, and if $\mathrm{cov}[X(s), X(t)]$ depends only on $|t-s|$.

Note: Since a Gaussian process is determined by its means and covariances, any **weakly stationary** Gaussian process is ________________.

Example (Ornstein-Uhlenbeck process):

- Let $\{X(t)\}$ be standard Brownian motion and for $\alpha > 0$, let

$$V(t) =$$

- Then $\{V(t), t \geq 0\}$ is the Ornstein-Uhlenbeck process, which is a common model for the velocity over time of an immersed particle.

$E[V(t)] =$

$\mathrm{cov}[V(t), V(t+s)] =$

- So $\{V(t)\}$ is

- Since Brownian motion is Gaussian, $\{V(t)\}$ is also Gaussian; thus $\{V(t)\}$ is:

<u>Example</u> (Nonsymmetric Random Telegraph):

- Suppose we alter the random telegraph signal process so that $E[X_0]$ is still 0, but the distribution of X_0 is not symmetric around 0.

Then $E[X(t)] =$

and $\text{cov}[X(t), X(t+s)] =$

Time Series Examples

Example (Autoregressive Process):

- Let $Z_0, Z_1, Z_2, \ldots$ be uncorrelated r.v.'s, each with mean zero and

  $\text{var}(Z_n) =$

  where $(1-\lambda^2) > 0$.
- Define $X_0 = Z_0$ and $X_n = \lambda X_{n-1} + Z_n$, $n \geq 1$.
- Then $\{X_n\}$ is a first-order autoregressive (AR-1) process.

- The state at time n is a multiple of the previous state, plus some random "noise."

Note $X_n =$

So $E[X_n] =$ and

$cov[X_n, X_{n+m}] =$

- So $\{X_n\}$ is

Example (Moving Average): Let $W_0, W_1, W_2, \ldots$ be uncorrelated r.v.'s with $E[W_n] = \mu$ and $var[W_n] = \sigma^2$ for $n \geq 0$. For some positive integer k, let

$X_n =$

- That is, X_n is the average of the most recent $(k+1)$ values of the W_i's (a moving average).

Clearly $E[X_n] =$ and

$\text{cov}[X_n, X_{n+m}] =$