8.4 Estimation Error Bounds

- Let $\varepsilon = |\hat{\theta} - \theta|$ be the error in estimation for an estimator $\hat{\theta}$ of θ.

Empirical Rule: Suppose $\hat{\theta}$ has an approximately normal sampling distribution with mean θ and variance $\sigma_{\hat{\theta}}^2$.

Then:

- These follow from basic normal probabilities.

Example 1: To estimate an unknown population mean μ, we take a random sample of size 49, and we calculate $\overline{Y}$. What is an approximate 95% bound on ε?

Note: If $\hat{\theta}$ does not have an approximately normal sampling distribution, we can still get a conservative bound on ε.

Chebyshev's Inequality says:

for $k > 0$.

So if $k = 2$, then

8.5 Confidence Intervals

- A confidence interval (or interval estimator) is an interval of numbers containing "reasonable values" for an unknown parameter (which we may generally label θ).
- We would like our interval estimator to:
  (1) have a high probability of containing the true value of θ
  (2) be relatively narrow (precise)
- A random interval $[\hat{\theta}_L, \hat{\theta}_u]$ is a $100(1-\alpha)\%$ CI for θ if:

- The number $1-\alpha$ is called the confidence coefficient.

- We often choose $1-\alpha$ to be large, like:

- We could also have one-sided CIs like $[\hat{\theta}_L, \infty)$ or $(-\infty, \hat{\theta}_u]$ where

$$P(\hat{\theta}_L \leq \theta) = 1-\alpha \quad \text{or} \quad P(\theta \leq \hat{\theta}_u) = 1-\alpha.$$

Pivotal Method

- A useful method for deriving confidence intervals is to use a <u>pivotal</u> <u>quantity</u>:
- A pivotal quantity
  (1) is a function of the sample data, the unknown target parameter, and <u>no</u> <u>other</u> unknown quantities.
  (2) has a distribution that does not depend on the target parameter.

<u>Example 1:</u> We will randomly sample $n=1$ observation from an exponential distribution with unknown mean θ. Find a formula for a 90% CI for θ.

If $Y \sim \text{expon}(\theta)$, then $f_Y(y) =$

- It is easily shown that the density of $U = \frac{Y}{\theta}$ is:

- Note $U = \frac{Y}{\theta}$ is clearly a pivotal quantity.

- We need to find two numbers a and b such that:

Picture:

- One idea: Set

Then:

Solve for a and b:

So:

- We want to isolate θ in the middle:

Example 2: We sample $n=1$ observation from a $\text{Unif}(0,\theta)$ distribution where θ is unknown. Find a 95% lower confidence bound for θ.

$Y \sim \text{Unif}(0,\theta)$. It can be shown that

$$\frac{Y}{\theta} \sim$$

Hence $\frac{Y}{\theta}$ is a pivotal quantity.

- We want some number a such that:

- If we observe $Y = 3.8$ from this uniform distribution, then our 95% lower confidence bound is:

- Hence we are 95% confident that θ is

Example 3: Let $X_1, \ldots, X_{10}$ be iid r.v.'s from an exponential distribution with unknown mean θ. Use the pivotal method to find a 95% CI for θ.

- Recall from Sec. 6.5: If $X_1, \ldots, X_n \overset{iid}{\sim} \text{expon}(\theta)$, then:

- Also from Sec. 6.5, if

Example 4: Let $Y_1, \ldots, Y_n$ be iid Unif $(0, \theta)$. Use the pivotal method to find a 95% upper confidence bound for θ.

-Recall the maximum, $Y_{(n)}$, has cdf:

- Hence consider $U =$

$F_U(u) =$

Example: If we have $n = 10$, with the maximum $Y_{(n)} = 5.7$, we are 95% confident that θ is at most: