Defn: If Z and W are independent, where $Z \sim N(0,1)$ and $W \sim \chi^2_\nu$, then $T = \dfrac{Z}{\sqrt{\frac{W}{\nu}}}$

has a t-distribution with ν degrees of freedom.

- The density of T is:

Proof: Use bivariate transformation technique with $U_1 = \dfrac{Z}{\sqrt{\frac{W}{\nu}}}$ and auxilary variable $U_2 =$

Then:

Exercise: Show that this reduces to the t density.

Lemma: If $W \sim \chi^2_\nu$, then $E\left(\frac{1}{W}\right) = \frac{1}{\nu - 2}$ (for $\nu > 2$).

Proof:

Theorem: If T has a t-distribution with ν d.f., then: $E(T) = 0$ (if $\nu > 1$) and $\text{var}(T) = \frac{\nu}{\nu - 2}$ (if $\nu > 2$).

Proof:

<u>Note:</u> For $\nu > 2$, $\text{var}(T) = \frac{\nu}{\nu-2} > 1 = \text{var}(Z)$.

- Comparing t density to a standard normal density, both are centered at _____ but the t-distribution has _______ variance.

<u>Picture:</u>

- As $\nu \longrightarrow \infty$, $\text{var}(T) \longrightarrow$

- For <u>smaller</u> d.f., the t-distribution is _____ spread out.

- As $\nu \to \infty$, the t-distribution approaches:

The t-distribution and the Sample Mean

Theorem: If $Y_1, \ldots, Y_n$ are independent $N(\mu, \sigma^2)$ r.v.'s, then

$$\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

Proof:

Example 1(c): Repeat 1(a), but this time assume $s = 26$ ml is the sample standard deviation.

The F-distribution

- Sometimes we have two random samples taken from two normal populations: $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$.
- We may wish to compare the two population variances, σ_1^2 and σ_2^2.
- A reasonable approach is to examine the sample variances, S_1^2 and S_2^2.

Defn: If W_1, W_2 are independent r.v.'s, $W_1 \sim \chi^2_{\nu_1}$ and $W_2 \sim \chi^2_{\nu_2}$, then $F = \dfrac{W_1/\nu_1}{W_2/\nu_2}$ has an F-distribution with ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom.

If $Y \sim F_{\nu_1, \nu_2}$, then its pdf is:

Proof: Let $U_1 = \dfrac{W_1/\nu_1}{W_2/\nu_2}$, $U_2 = W_2/\nu_2$. Use bivariate transformation technique to eventually obtain marginal pdf of U_1.

Note: If $\nu_2 > 2$, then for $Y \sim F_{\nu_1, \nu_2}$:

$E(Y) =$

- The mean of an F r.v. depends only on the denominator d.f.!
- Also, if $\nu_2 > 4$, $\text{var}(Y) =$
- Can be shown using $\text{var}(Y) = E(Y^2) - [E(Y)]^2$ and properties of χ^2 r.v.'s.

Picture of F density:

Properties of F r.v.'s:

① If $Y \sim F_{\nu_1, \nu_2}$, then $\frac{1}{Y} \sim F_{\nu_2, \nu_1}$.

Proof:

② If $Y \sim t_\nu$, then $Y^2 \sim F_{1,\nu}$.

Proof:

③ If $Y \sim F_{\nu_1,\nu_2}$, then $\dfrac{\left(\frac{\nu_1}{\nu_2}\right) Y}{1+\left(\frac{\nu_1}{\nu_2}\right) Y} \sim \text{Beta}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$.

Theorem: For two independent random samples of sizes n_1 and n_2 from normal populations with variances σ_1^2 and σ_2^2, the statistic

Proof:

Example 3: Two independent samples of sizes 16 and 21 are taken from two normal populations with $\sigma_1^2 = 3$ and $\sigma_2^2 = 5$. What is the probability that S_2^2 is at least twice as large as S_1^2?