STAT 512 - Mathematical Statistics

Chapter 6: Functions of Random Variables

Recall: If Y is a random variable, then its cumulative distribution function (or c.d.f.) is defined as:

$F_Y(y) =$ for any number y.

If Y is a continuous r.v., then the probability density function (or p.d.f.) of Y is:

$f_Y(y) =$

- Often we may be interested in another r.v., say U, where U is a function of Y, i.e., $U \equiv U(Y)$.
- Or we might be interested in a function of several random variables, e.g., $U(Y_1, Y_2, \ldots, Y_n)$.

Example: If Y_1, Y_2, Y_3, Y_4 are a random sample from a population, we may be interested in the distribution of:

- We will study three methods for determining the distribution of a function of a random variable:

(1) The method of cdf's
(2) The method of transformations
(3) The method of mgf's

6.3 The Method of cdf's

- For a function U of a continuous r.v. Y with a known density $f(y)$, we can often find the distribution of U using the definition of the cdf:

Example 1 (Univariate)

Y = amount of sugar produced per day (in tons)

Suppose $f(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

Let the profit $U = 3Y - 1$ (in hundreds of \$)

Then $F_U(u) =$

Note Y can range from

Hence $U = 3Y - 1$ can range from

So $F_U(u) =$

and $f_U(u) =$

Example 2: Let Y have the pdf:

$$f(y) = \begin{cases} 6y(1-y), & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $U = Y^3$.

Bivariate situation

- Let $U = h(Y_1, Y_2)$ be a function of r.v.'s Y_1 and Y_2 that have joint density $f(y_1, y_2)$.

  We must:

* Find the values (y_1, y_2) such that $U \le u$.
* Integrate $f(y_1, y_2)$ over this region to obtain $P(U \le u) = F_U(u)$.
* Differentiate $F_U(u)$ to obtain $f_U(u)$.

Example 1: Let Y_1 = the amount of gasoline stocked at the beginning of the week. Let Y_2 = the amount of gasoline sold during the week. The joint density of Y_1 and Y_2 is:

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density of $U = Y_1 - Y_2$, the amount of gasoline remaining at the end of the week.

Picture:

- Easier to integrate lower triangular region.

So $F_u(u) =$

Exercise: Show the *expected* amount of gasoline remaining at the end of the week is $3/8$.

Example 2: Suppose the joint density of Y_1 and Y_2 is:

$$f(y_1, y_2) = \begin{cases} 6e^{-3y_1 - 2y_2} & \text{if } y_1 > 0,\ y_2 > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = Y_1 + Y_2$.

Picture:

Then $F_u(u) =$

Transforming a Uniform (0,1) r.v.

- Recall $U \sim \text{Unif}(0,1)$ has cdf:

$$F_U(u) = \begin{cases} 0, & u < 0 \\ u, & 0 \le u \le 1 \\ 1, & u > 1 \end{cases}$$

- We can find a transformation $Y = G(U)$ such that $G(U)$ has a specified cdf $F_Y(y)$, as long as the inverse $F^{-1}(y)$ is unique and well-defined.

<u>Example:</u> Transform U into $Y \sim \text{expon}(\beta)$.

Note: $F_Y(y) =$

<u>Application</u>: To simulate values $Y_1, \ldots, Y_n$ from an Expon(β) distribution, just generate $U_1, \ldots, U_n$ from a Unif$(0,1)$ and let $Y_i = -\beta \ln(1 - U_i)$, $i = 1, \ldots, n$.

Example 3: Let Y_1 and Y_2 have joint pdf

$$f(y_1, y_2) = \begin{cases} 5y_1 e^{-y_1 y_2} & \text{for } 0.2 < y_1 < 0.4,\ y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $U = Y_1 Y_2$.

Picture: