3.9 Moments and Moment-Generating Functions

Defn: The k-th moment about the origin of a r.v. Y (also simply called the k-th moment of Y) is defined as:

Examples of moments:

- The first moment, μ_1', is the ________ of Y.
- We know $\mu_2' - [\mu_1']^2$ is the ________ of Y.
- The third moment μ_3' is related to the skewness (lack of symmetry) of a distribution.
- The fourth moment μ_4' is related to the kurtosis (peakedness) of a distribution.

Note: The expression $E[(Y-\mu)^k]$ is sometimes called the k-th moment about the mean (or k-th central moment) of Y and is denoted by .

Defn (Moment-generating function):
The moment-generating function (or mgf) of a r.v. Y is denoted by $m_Y(t)$ and defined to be :

- The mgf for Y exists if there exists some $b > 0$ such that

- If $E[e^{tY}]$ does not exist in any open neighborhood around $t=0$, then $m_Y(t)$ does not exist.
- If the mgf exists, it characterizes all the moments of Y,

$$E(Y^k), \quad k=1,2,3,\ldots$$

<u>Theorem</u>: If $m_Y(t)$ exists, then for any integer $k \geq 1$,

<u>Examples</u>: To get the first moment $E(Y)$, take the derivative (with respect to t) $m_Y'(t)$ and plug in $t=0$.

- To get the second moment $E(Y^2)$, take the 2nd derivative $m_Y''(t)$ and plug in $t=0$.
- To prove the theorem, recall:

The Maclaurin series expansion of an infinitely differentiable function $f(y)$ is given by:

-We can easily verify that the Maclaurin series expansion of $f(y) = e^{ty}$ is:

So $m_y(t) =$

So plugging in $t=0$,

- This proof requires interchanging derivatives and infinite sums, which is justified if $m_Y(t)$ exists.

<u>Example 1:</u> (Recalls Petersburg paradox):

Let a r.v. Y have probability function

$$p(y) = \left(\frac{1}{2}\right)^y \quad \text{for } y = 1, 2, 3, \ldots$$

Find $m_Y(t)$ and then $E(Y)$.

Example 2: Let Y have probability function

$$p(y) = \frac{y}{10} \quad \text{for} \quad y = 1, 2, 3, 4.$$

Find $m_Y(t)$ and $E(Y)$.

- For Example 2, it is easier to find $E(Y)$ directly using the definition.

Example 3 (Variance of a Poisson r.v.):

Let $Y \sim \text{Pois}(\lambda)$ with probability function

$$p(y) = \frac{\lambda^y}{y!} e^{-\lambda} \quad \text{for } y = 0, 1, 2, \ldots$$

Find the mgf and variance of Y.

Lemma: For any number x,

Thus

Then

We have proved that $E(Y) = \lambda$ for $Y \sim \text{Pois}(\lambda)$, so

<u>Example 4:</u> The mean and variance of a Bin(n, p) r.v. can also be found using the binomial mgf:

which may be derived by considering the binomial expansion of $(pe^t + q)^n$.

<u>Example 5:</u> (Mean and variance of a negative binomial r.v.):

Let $Y \sim NB(r, p)$. Find $m_Y(t)$, $E(Y)$, and $V(Y)$.

<u>Lemma:</u> For any integer $r \geq 1$,

<u>Proof:</u>

Hence

Letting $x =$ and $j =$ yields the lemma.

Now $m_Y(t) = E[e^{tY}] =$

To get the variance, after much tedious calculus:

<u>Corollary</u> (Geometric variance):

If $Y \sim geom(p)$, then $V(Y) =$

- An important fact is that a mgf completely characterizes a probability distribution.
- If $m_Y(t)$ exists for distribution $p(y)$, then it is the unique mgf for Y.

- If two r.v.'s have the same mgf, then they <u>must</u> have the same distribution.
- So we can often use the mgf to identify the distribution of a r.v.

<u>Example 6</u>: If Y is a r.v. with mgf $m_Y(t) = e^{7.1e^t - 7.1}$, then what is the distribution of Y?

<u>Example 7</u>: If Y has mgf $m_Y(t) = \frac{e^t}{2-e^t}$, then what is the distribution of Y?