Outline of Proof of independence of $\overline{Y}$ and S^2, when $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$:

- Assume WLOG that $Y_1, \dots, Y_n \overset{iid}{\sim} N(0,1)$. The proof also holds for any arbitrary μ and positive σ^2, using similar arguments.

Note $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \frac{1}{n-1} \left[\{Y_1 - \overline{Y}\}^2 + \sum_{i=2}^{n} (Y_i - \overline{Y})^2 \right]$

$$= \frac{1}{n-1} \left[\left\{ \sum_{i=1}^{n} (Y_i - \overline{Y}) - \sum_{i=2}^{n} (Y_i - \overline{Y}) \right\}^2 + \sum_{i=2}^{n} (Y_i - \overline{Y})^2 \right]$$

$$= \frac{1}{n-1} \left[\left\{ \sum_{i=2}^{n} (Y_i - \overline{Y}) \right\}^2 + \sum_{i=2}^{n} (Y_i - \overline{Y})^2 \right]$$

(since $\sum_{i=1}^{n} (Y_i - \overline{Y}) = \sum_{i=1}^{n} Y_i - n\overline{Y} = n\overline{Y} - n\overline{Y} = 0$).

Hence S^2 is a function of $(Y_2 - \overline{Y}), (Y_3 - \overline{Y}), \dots, (Y_n - \overline{Y})$.

Use the multivariate transformation:

$$\begin{aligned} U_1 &= \overline{Y} \\ U_2 &= Y_2 - \overline{Y} \\ &\vdots \\ U_n &= Y_n - \overline{Y} \end{aligned}$$

It can be shown that the Jacobian of the transformation is n.

Now, $f_{\underset{\sim}{Y}}(y_1, \dots, y_n) = \prod_{i=1}^{n} f_{\underset{\sim}{Y}}(y_i) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \sum_{i=1}^{n} y_i^2}$

Note: $\sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \sum_{i=1}^{n} Y_i^2 - 2\overline{Y} \sum_{i=1}^{n} Y_i + n\overline{Y}^2 = \sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2$

$$\Rightarrow \sum_{i=1}^{n} Y_i^2 = \sum_{i=1}^{n} (Y_i - \overline{Y})^2 + n\overline{Y}^2$$

$$= \sum_{i=2}^{n} (Y_i - \overline{Y})^2 + (Y_1 - \overline{Y})^2 + n\overline{Y}^2$$

$$= \sum_{i=2}^{n} u_i^{\,2} + \left(\sum_{i=2}^{n} u_i\right)^2 + n\,u_1^{\,2}$$

since $(Y_1-\bar{Y})^2 = (\bar{Y}-Y_1)^2 = \left(\bar{Y} - \sum_{i=1}^{n} Y_i + \sum_{i=2}^{n} Y_i\right)^2$

$$= \left(\bar{Y} - n\bar{Y} + \sum_{i=2}^{n} Y_i\right) = \left[\sum_{i=2}^{n} Y_i - (n-1)\bar{Y}\right]^2$$

$$= \left[\sum_{i=2}^{n} (Y_i - \bar{Y})\right]^2 = \left(\sum_{i=2} u_i\right)^2$$

So by the multivariate method of transformations,

$$f_{\underset{\sim}{u}}(u_1,\dots,u_n) = f_{\underset{\sim}{y}}(y_1,\dots,y_n)\;|J|$$

$$= \frac{1}{(2\pi)^{n/2}}\, e^{-\frac{1}{2}\left[\sum_{i=2}^{n} u_i^2 + \left(\sum_{i=2}^{n} u_i\right)^2\right]}\, e^{-\frac{1}{2} n u_1^2}\,[n], \text{ for}$$

$$-\infty < u_1 < \infty,\ -\infty < u_2 < \infty, \dots, -\infty < u_n < \infty.$$

$\Rightarrow$ The joint pdf factors into a piece depending only on u_1 and a piece depending only on $(u_2,\dots,u_n)$

$\Rightarrow$ u_1 and $(u_2,\dots,u_n)$ are independent.

$\Rightarrow$ $\bar{Y}$ and $\left((Y_2-\bar{Y}),\dots(Y_n-\bar{Y})\right)$ are independent.

$\Rightarrow$ $\bar{Y}$ and S^2 are independent.