STAT 730: Homework 3

1. **An important application of two theorems from Chapter 3 to univariate regression**. Consider the usual regression model from STAT 704: $\mathbf{y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\mathcal{I}}_n)$ where $\mathbf{X}(n\times p)$ is full rank, $n > p$, and $\boldsymbol{\beta}(p\times 1)$ are regression effects. The first column in $\mathbf{X}$ is $\mathbf{x}_{(1)} = \mathbf{1}_n$. Let $\mathbf{X} = [\mathbf{X}_1(n\times(p-k))\mathbf{X}_2(n\times k)]$ and $\boldsymbol{\beta}' = (\boldsymbol{\beta}_1((p-k)\times 1)', \boldsymbol{\beta}_2(k\times 1)')$. We have two models, reduced (R) and full (F):

$$R : \mathbf{y} \sim N_n(\mathbf{X}_1\boldsymbol{\beta}_1, \sigma^2\boldsymbol{\mathcal{I}}_n), \quad F : \mathbf{y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\mathcal{I}}_n).$$

Finally let $\mathbf{P}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, $\mathbf{P}_{\mathbf{X}_1} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$, and $\mathbf{P}_{\mathbf{1}_n} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$ We will construct the F-statistic for testing $H_0 : \boldsymbol{\beta}_2 = \mathbf{0}$ in the full model.

   (a) Show $SSR_d = \mathbf{y}'[\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_1}]\mathbf{y}$ has a scaled χ^2 distribution under H_0 using Cochran's theorem. Hint: first show $SSR_d = (\mathbf{y} - \mathbf{X}_1\boldsymbol{\beta}_1)'[\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_1}](\mathbf{y} - \mathbf{X}_1\boldsymbol{\beta}_1)$ and argue that $(\mathbf{y} - \mathbf{X}_1\boldsymbol{\beta}_1)$ is a d.m. under H_0.

   (b) Show $SSR_d = SSR_f - SSR_r$ where $SSR_r = \mathbf{y}'[\mathbf{P}_{\mathbf{X}_1} - \mathbf{P}_{\mathbf{1}_n}]\mathbf{y}$ and $SSR_f = \mathbf{y}'[\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{1}_n}]\mathbf{y}$. This is the difference in SSR from the full and reduced models (it is also the difference in the SSE). To see this, e.g., $SSR_f = \sum_{i=1}^n(\hat{y}_i - \bar{y})^2 = ||\hat{\mathbf{y}} - \mathbf{1}_n\bar{y}||^2 = ||\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{1}_n\bar{y}||^2 = ||(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{1}_n})\mathbf{y}||^2 = \mathbf{y}'[\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{1}_n}]\mathbf{y}$, using $\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{1}_n}$ idempotent & symmetric.

   (c) Similarly, show $SSE_f = \mathbf{y}'[\boldsymbol{\mathcal{I}}_n - \mathbf{P}_{\mathbf{X}}]\mathbf{y}$ has a scaled χ^2 distribution under H_0 using Cochran's theorem.

   (d) Show SSR_d indep. of SSE_f using the "partitioning sums of squares", i.e. Craig's theorem.

   (e) Finally, show $F^* = \frac{SSR_d/k}{SSE_f/(n-p)} \sim F_{k,n-p}$ under H_0.

2. Using the previous "big model / little model" result, show that the F-test in the usual ANOVA table for univariate multiple regression has a $F_{p-1,n-p}$ distribution.

3. Show $E(\mathbf{x}'\mathbf{A}\mathbf{x}) = \text{tr}\mathbf{A}\boldsymbol{\Sigma} + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ where $\mathbf{x} \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is random and $\mathbf{A}$ is conformable. Hint: $E\{\text{tr}(\cdot)\} = \text{tr}\{E(\cdot)\}$ and $\text{tr}\mathbf{AB} = \text{tr}\mathbf{BA}$. Now find $E(SSR_f)$ when the full model holds; under H_a in the previous problem.

4. Show $\frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}$ and $\frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}')\mathbf{x}$.

5. Let $\mathbf{A} = [a_{ij}]$, all elements distinct. Show $\frac{\partial |\mathbf{A}|}{\partial a_{ij}} = A_{ij}$ where A_{ij} is the ijth cofactor of $\mathbf{A}$. Hint: use (I) on p. 457 plus a careful argument. Note: the result for symmetric $\mathbf{A}$ is considerably more difficult.

6. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \overset{iid}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ under the priors $\boldsymbol{\mu} \sim N_p(\mathbf{m}, \mathbf{V})$ indep. $\boldsymbol{\Sigma}^{-1} \sim W_p(\mathbf{A}, a)$. Show

$$\boldsymbol{\mu}|\boldsymbol{\Sigma}, \mathbf{X} \sim N_p([n\boldsymbol{\Sigma}^{-1} + \mathbf{V}^{-1}]^{-1}[n\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}} + \mathbf{V}^{-1}\mathbf{m}], [n\boldsymbol{\Sigma}^{-1} + \mathbf{V}^{-1}]^{-1}),$$

and

$$\boldsymbol{\Sigma}^{-1}|\boldsymbol{\mu}, \mathbf{X} \sim W_p\left(\left[\mathbf{A}^{-1} + \sum_{i=1}^n(\mathbf{x}_i - \boldsymbol{\mu})(\mathbf{x}_i - \boldsymbol{\mu})'\right]^{-1}, a + n\right).$$

7. MKB 4.2.5.

8. MKB 4.2.12.