STAT 730: Homework 2

1. **MLE of multivariate normal**. We will fill in details of a maximization theorem for $\boldsymbol{\Sigma}$ in Chapter 4 (p. 105). Let $\mathbf{A} > 0$ and $\mathbf{B} > 0$ be symmetric in all that follows.

   (a) Show $\mathbf{A} > 0 \Leftrightarrow$ all e-values of $\mathbf{A}$ are positive.

   (b) Show $\mathbf{A}^{-1} > 0$.

   (c) Show $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$ has the same e-values as $\mathbf{A}^{-1}\mathbf{B}$. Hint: let $\mathbf{v}$ be an e-vector of $\mathbf{A}^{-1}\mathbf{B}$ and show $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{v}$ is an e-vector of $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$.

   (d) Show the e-values of $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$ are all positive. Hint: $\mathbf{B} > 0$.

   Thus the e-values of $\mathbf{A}^{-1}\mathbf{B}$ are all positive by (c) and (d); (a) and (b) are just for fun.

2. **Characeristic function of normal**: Let $x \sim N(0,1)$.

   (a) Show $E(x^p) = 0$ if p odd and $E(x^p) = (p-1)(p-3)(p-5)\cdots 1$ if p is even. Hint: look at $\frac{1}{2}E(x^p) = I = \int_0^\infty x^p \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}dx$ and make the change of variables $y = x^2$. Then use properties of $\Gamma(\cdot)$.

   (b) Now use the Taylor's expansion $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ in $E\{e^{itx}\}$, take expectation of each term, and simplify. This works because $\int_a^b ix^p f(x)dx = i\int_a^b x^p f(x)dx$ by definition and the Taylor's expansion holds for complex arguments. You should have $\phi_x(t) = e^{-t^2/2}$.

   (c) Let $y \sim N(\mu, \sigma^2)$. Show $\phi_y(t) = e^{it\mu - \sigma^2 t^2/2}$. Hint: write $y = \sigma x + \mu$.

3. **Details of delta method proof**. Consider $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^q$ where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_q(\mathbf{x}) \end{bmatrix}.$$

   For the ith component of $\mathbf{f}$, the multivariate Taylor's theorem gives us

$$f_i(\mathbf{x}) = f_i(\mathbf{x}_0) + [Df_i(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0}](\mathbf{x} - \mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)'\mathbf{H}_i(\mathbf{x})(\mathbf{x} - \mathbf{x}_0),$$

   where $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{H}_i(\mathbf{x}) = \mathbf{0}$. Here, $Df_i(\mathbf{x})$ is the row-vector of first-partials $[\frac{\partial f_i}{\partial x_1} \cdots \frac{\partial f_i}{\partial x_p}]$.

   (a) Use Cauchy-Schwartz to show the absolute value of last term in the expansion is bounded by $||\mathbf{x} - \mathbf{x}_0||\delta_i(\mathbf{x} - \mathbf{x}_0)$ where $\lim_{\mathbf{x}\to\mathbf{x}_0} \delta_i(\mathbf{x} - \mathbf{x}_0) = 0$.

   (b) In the proof of the Delta method, further argue that $\sqrt{n}||\mathbf{t} - \boldsymbol{\mu}|| = O_p(1)$ and $||\boldsymbol{\delta}(\mathbf{t} - \boldsymbol{\mu})|| = o_p(1)$.

4. Let $y_1, \ldots, y_n \stackrel{iid}{\sim} N(\boldsymbol{\mu}, \sigma^2)$, i.e. $\mathbf{y} \sim N_n(\mu\mathbf{1}_n, \sigma^2\boldsymbol{\mathcal{I}}_n)$. Show $\bar{y} = \frac{1}{n}\mathbf{1}_n'\mathbf{y}$ is indep. of $s^2 = \frac{1}{n}\sum_{i=1}^n (y_i - \bar{y})^2$ using the property of multivariate normals where zero covariance implies independence. Do this by looking at $\mathbf{A}\mathbf{y}$ where $\mathbf{A} = \begin{bmatrix} \mathbf{1}_n' \\ \boldsymbol{\mathcal{I}}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n' \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}$ and making the appropriate argument.

5. Let $\phi_{\mathbf{x}}(\mathbf{t})$ be the c.f. of $\mathbf{x}$. Find the c.f. of $\mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are conformable and fixed, in terms of $\phi_{\mathbf{x}}(\mathbf{t})$.

6. MKB 3.3.5.

7. MKB 4.2.3.