STAT 506: Randomized complete block designs

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STAT 506: Introduction to Experimental Design

Randomized complete block designs

Subjects placed into homogeneous groups, called *blocks*. All treatment combinations assigned randomly to subjects within blocks.

Example: executives exposed to one of three methods (treatment, $i = 1$ utility method, $i = 2$ worry method, $i = 3$ comparison method) of quantifying maximum risk premium they would be willing to pay to avoid uncertainty in a business decision. Response is "degree of confidence" in the method on a scale from 0 (no confidence) to 20 (complete confidence). It is thought that confidence is related to age, so the subjects are blocked according to age ($j = 1, 2, 3, 4, 5$ from oldest to youngest). $N = 15$ subjects are recruited, with three subjects in each of the 5 age categories. Within each age category, the three subjects are randomly given one of the three treatments.

RCB designs, comments

- With thoughtful blocking, can provide more precise results than completely randomized design.
- There is only one replication for each pairing of treatment and block; need to assume no interaction between treatments and blocks to obtain estimate of σ^2.
- The blocking variable is observational, not experimental. Cannot infer causal relationship. Not a problem though...usually only care about treatments.
- Blocking works by making σ^2 smaller; "block effects" are variability in the data that normally go into σ^2 when they are ignored.

Model and inference

One observation per block/treatment combination gives $N = ab$. Need to fit additive model to get SSE > 0

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}.$$

Estimates obtained via LS as usual,

$$Q(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^{a} \sum_{j=1}^{b} (Y_{ij} - [\mu + \alpha_i + \beta_j])^2$$

minimized subject to $\alpha_a = \beta_b = 0$ (SAS and R's default) or $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$ (`cfcdae`'s and Minitab's default). Your book uses the notation g treatments and r blocks; I'm keeping a and b as before, because even though blocks are not treatments, the model is analyzed as an additive twoway ANOVA model.

ANOVA table

Source	SS	df	MS	F	p-value
A	$SSA = b\sum_{i=1}^{a}(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})^2$	$a-1$	$\frac{SSA}{a-1}$	$\frac{MSA}{MSE}$	p_1
B	$SSB = a\sum_{j=1}^{b}(\bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet})^2$	$b-1$	$\frac{SSB}{b-1}$	$\frac{MSB}{MSE}$	p_2
Error	$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{i\bullet} - \bar{Y}_{\bullet j} + \bar{Y}_{\bullet\bullet})^2$	$(a-1)(b-1)$	$\frac{SSE}{(a-1)(b-1)}$		
Total	$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{\bullet\bullet})^2$	$ab-1$			

Here, $p_1 = P\{F(a-1,(a-1)(b-1)) > \frac{\text{MSA}}{\text{MSE}}\}$ tests $H_0 : \alpha_1 = \cdots = \alpha_a = 0$ (no blocking effect) and $p_2 = P\{F(b-1,(a-1)(b-1)) > \frac{\text{MSB}}{\text{MSE}}\}$ tests $H_0 : \beta_1 = \cdots = \beta_b = 0$ (no treatment effect). These appear in SAS as Type III tests.

If reject $H_0 : \beta_j = 0$, then obtain inferences in treatment effects as usual, e.g. `pairwise(f,treatment)`.

Diagnostics

1. Interaction (also called spaghetti or profile) plots of the y_{ij} vs. treatment j, connected by block i are useful. Should be somewhat parallel if additive model is okay, but there is a lot of sampling variability here as $\hat{\mu}_{ij} = y_{ij}$.
2. Standard R diagnostic panel: e_{ij} vs. $\hat{y}_{ij}$, normal probability plot of the $\{e_{ij}\}$, etc. Can also look at e_{ij} vs. either i or j, should show constant variance within blocks and treatments.
3. Tukey's test for additivity.

Tukey's test for additivity

Reduced model is additive $Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$. Full model is

$$Y_{ij} = \mu + \alpha_i + \beta_j + D\alpha_i\beta_j + \epsilon_{ij}.$$

This is more restrictive than using a general interaction $(\alpha\beta)_{ij}$, leaves df to estimate error.

$$\hat{D} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})(\bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet})}{\sum_{i=1}^{a}(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})^2 \sum_{j=1}^{b}(\bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet})^2}.$$

$$SSAB^* = \sum_{i=1}^{a}\sum_{j=1}^{b} \hat{D}^2(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})^2(\bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet})^2,$$

and SSTO=SSA+SSB+SSAB*+SSE*.

$$F* = \frac{SSAB^*}{SSE^*/(ab - a - b)} \sim F(1, ab - a - b),$$

if $H_0 : D = 0$ is true.

I placed a script on the STAT 506 webpage to compute the p-value and D for you.

Confidence ratings, pp. 895-896

```
conf=c(1,2,7,6,12,5,8,9,13,14,8,14,16,18,17)
method=factor(c(rep("utility",5),rep("worry",5),rep("comparison",5)))
age=factor(rep(1:5,3))
d=data.frame(conf,method,age)
par(mfrow=c(1,1))
with(d,interactplot(age,method,conf)) # parallel?
with(d,interactplot(method,age,conf)) # parallel?
source("http://people.stat.sc.edu/hansont/stat506/tukey.R")
tukeys.add.test(d$conf,d$age,d$method) # accept additive model ok

f=lm(conf~method+age,data=d)
Anova(f,type=3)
pairwise(f,method)
lines(pairwise(f,method))
par(mfrow=c(2,2))
plot(f)
```

Sample size and power

Russ Lenth's power applet uses

$$sd(A) = \sqrt{\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i^2}, \quad sd(B) = \sqrt{\frac{1}{b-1}\sum_{j=1}^{a}\beta_j^2},$$

assuming $\sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = 0$. Same as

$$sd(A) = \sqrt{\frac{1}{a-1}\sum_{i=1}^{a}(\bar{\mu}_{i\bullet} - \bar{\mu}_{\bullet\bullet})^2}, \quad sd(B) = \sum_{j=1}^{b}\sqrt{(\bar{\mu}_{\bullet j} - \bar{\mu}_{\bullet\bullet})^2},$$

First one is `SD[Block]` and second `SD[treatment]` in the dialogue box. Also need an estimate of σ^2. The GUI will give you estimates of power to reject $H_0 : \alpha_i = 0$ (which we don't care about) and $H_0 : \beta_j = 0$ (which we care about).

Example

Say you are designing a RCBD where it $\sigma \approx 2$, the $b = 3$ treatment effects are thought to be $\bar{\mu}_{\bullet 1} = 15$, $\bar{\mu}_{\bullet 2} = 15$, and $\bar{\mu}_{\bullet 3} = 18$. Then $\bar{\mu}_{\bullet\bullet} = \frac{1}{3}(15 + 15 + 18) = 16$ and $sd(B) = \sqrt{\frac{1}{2}(1 + 1 + 4)} = \sqrt{3} \approx 1.73$.

Variability in blocking effects is thought to vary over a range of 6 units. Then, crudely $sd(A) = \frac{\text{range}}{4} = \frac{6}{4} = 1.5$.

Russ Lenth's dialogue box then shows us $a = 10$ blocks gives a power of about 90%.

Factorial treatments in a RCBD: dental pain

Anesthesiologist studied effects of acupuncture and codeine on dental pain in $N = 32$ male subjects. Pain relief scores (higher = less pain) recorded; two factors A (placebo or codeine) and B (inactive vs. active acupuncture site). Subjects blocked on initial pain tolerance...why?

```
library(cfcdae); library(car); library(lsmeans)

pain=c(0.0,0.6,0.5,1.2,0.3,0.7,0.6,1.3,0.4,0.8,0.8,1.6,0.4,0.9,0.7,1.5,
       0.6,1.5,1.0,1.9,0.9,1.6,1.4,2.3,1.0,1.7,1.8,2.1,1.2,1.6,1.7,2.4)
tol=factor(rep(1:8,each=4)) # pain tolerance
drug=factor(c(1,1,2,2,1,1,2,2,1,1,2,2,1,1,2,2,1,1,2,2,1,1,2,2,1,1,2,2))
acupuncture=factor(rep(1:2,16))
d=data.frame(pain,tol,drug,acupuncture)
levels(d$drug)=c("placebo","codeine")
levels(d$acupuncture)=c("inactive","active")

with(d,interactplot(acupuncture:drug,tol,pain))
source("http://people.stat.sc.edu/hansont/stat506/tukey.R")
tukeys.add.test(d$pain,d$tol,d$drug:d$acupuncture)

f=lm(pain~tol+drug*acupuncture,data=d)
```

Let's keep going...