STAT 703/J703
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-Lecture 14-

Instructor: Brian Habing
Department of Statistics
LeConte 203
Telephone: 803-777-3578
E-mail: habing@stat.sc.edu

Today

- 9.3: Proof of Neyman-Pearson Lemma
- Another N-P Example
- 9.5: Generalized Likelihood Ratio Test

Neyman-Pearson Lemma: If the likelihood ratio test that rejects H_0 in favor of H_A when

$$\lambda = \frac{f_0(x)}{f_A(x)} \leq c$$

has significance level α,

then any other test having significance level at most α has power less than or equal to the power of the likelihood ratio test. (I.e., the LRT has highest power among tests with significance level α).

Proof: Let C be the set of possible samples such that $\lambda = \frac{f_0(\underline{x})}{f_A(\underline{x})} \leq c$

Let C' be the set of possible samples such that $\lambda = \frac{f_0(\underline{x})}{f_A(\underline{x})} \leq c$

Assume there exists some other test with significance level a and it has corresponding regions D and D'

To begin, note that we have

$$\alpha = \int_C f_0(\underline{x}) = \int_D f_0(\underline{x})$$

….

We also use the knowledge that

$$f_0(\underline{x}) \leq c f_A(\underline{x}) \text{ for all } \underline{x} \in C$$
$$\text{and thus also} \in C \cap D'$$

….

So that we have

$$\int_C f_A(\underline{x}) \geq \int_D f_A(\underline{x})$$

<u>Another Example:</u> The number of down time incidences is supposed to average fewer than 2 per month. Six machines are examined for 2 months each and result in 0, 3, 6, 1, 3, 5 down-time incidences respectively. Conduct a test of the appropriate null and alternate hypotheses to see if you should complain.

<u>9.5 Generalized Likelihood Ratio Tests</u>

The Neyman-Pearson likelihood ratio test is most powerful for simple vs. simple.

Here, we generalize to composite hypotheses. The generalized LRT is not necessarily optimal, but works well for situations where no optimal test exists.

Observe $X_1, \ldots, X_n$ from $f(x|\theta)$.

Test H_0: $\theta \in \omega_0$ $\quad \omega_0 \subset \Omega$

vs. H_A: $\theta \in \omega_1$ $\quad (\omega_0 \cup \omega_1 = \Omega)$.

Use the generalized LR statistic.

$$\Lambda = \frac{\max_{\theta \in \omega_0} \left(lik(\theta)\right)}{\max_{\theta \in \Omega} \left(lik(\theta)\right)} = \frac{\max_{\theta \in \omega_0} \prod_{i=1}^{n} f(x_i \mid \theta)}{\max_{\theta \in \Omega} \prod_{i=1}^{n} f(x_i \mid \theta)}$$

Reject H_0 if $\Lambda \le \lambda_0$, where λ_0 is $P(\text{rej. } H_0 | \theta \in \omega_0) = \alpha$.

(Note: If H_0 holds, $\Lambda=1$. If H_A holds, $\Lambda<1$, small).

Use this to construct the test, i.e. find rejection regions in terms of simple statistics (similar to N-P lemma).

Theorem A: Under smoothness conditions on the pdf, the null distribution of $-2\ln\Lambda$ has an approximate chi-square distribution with d.f.=dimW-dimω_0 for large n.

$N(\mu,\sigma^2)$, both unknown

$\Rightarrow$ df = 2-1 = 1.