Solutions to the Practice Problems for MoM and Maximum Likelihood

1) Consider a random sample x_1, … x_n from a distribution with pdf

$$f(x) = (\theta+1)(1-x)^{\theta} \quad \text{for } 0<x<1$$

a) Find the MoM estimator for θ.

$$\mu = \int_0^1 x(\theta+1)(1-x)^{\theta}dx \quad \text{Let y=1-x, so that x=1-y, and dx=-dy}$$

$$= -\int_{x=0}^{x=1}(1-y)(\theta+1)y^{\theta}dy = -(\theta+1)\int_{x=0}^{x=1}(y^{\theta}-y^{\theta+1})dy = -(\theta+1)\left(\frac{y^{\theta+1}}{\theta+1}-\frac{y^{\theta+2}}{\theta+2}\right|_{x=0}^{x=1}$$

$$= -(\theta+1)\left(\frac{y^{\theta+1}}{\theta+1}-\frac{y^{\theta+2}}{\theta+2}\right|_{y=1}^{y=0} = (\theta+1)\left(\frac{y^{\theta+1}}{\theta+1}-\frac{y^{\theta+2}}{\theta+2}\right|_{y=0}^{y=1} = (\theta+1)\left(\frac{1}{\theta+1}-\frac{1}{\theta+2}\right) = \frac{1}{(\theta+2)}$$

Or, notice that this is a beta distribution with α=1 and β=θ+1.

So $\bar{x} = \dfrac{1}{\hat{\theta}_{mom}+2} \Rightarrow \hat{\theta}_{mom}+2 = \dfrac{1}{\bar{x}} \Rightarrow \hat{\theta}_{mom} = \dfrac{1}{\bar{x}}-2$

b) Find the MLE estimator for θ.

$$lik(\theta\,|\,x_1,...x_n) = f(x_1,...x_n\,|\,\theta) = \prod_{i=1}^{n} f(x_i\,|\,\theta) = \prod_{i=1}^{n}(\theta+1)(1-x_i)^{\theta}$$

$$\log lik(\theta\,|\,x_1,...x_n) = \sum_{i=1}^{n}[\log(\theta+1)+\theta\log(1-x_i)] = n\log(\theta+1)+\theta\sum_{i=1}^{n}\log(1-x_i)$$

$$\frac{\partial}{\partial\theta}\log lik(\theta\,|\,x_1,...x_n) = \frac{n}{\theta+1}+\sum_{i=1}^{n}\log(1-x_i) = 0$$

$$\Rightarrow \hat{\theta}_{mle} = \frac{-n}{\sum_{i=1}^{n}\log(1-x_i)}-1$$

2) In many cases a process will have a minimum value > 0 so that using a distribution like the exponential, chi-squared, F, or gamma doesn't make much sense. In this case the distribution can be shifted to the right. Consider the shifted exponential with pdf

$$f(x) = \frac{1}{\theta} e^{-(x-a)/\theta} \quad \text{for } x>a$$

and the data set 16.2,12.4, 6.0, 8.4, 6.8, 9.1, 6.6, 6.0, 10.7, 5.8.

a) Show that the mean of the shifted exponential is θ+a and the variance is θ^2.

$$\mu = \int_a^\infty x f(x)dx = \int_a^\infty x \frac{1}{\theta} e^{-(x-a)/\theta} dx$$ let y=x-a so that x=y+a and dx=dy and we get

$$= \int_{x=a}^{x=\infty} (y+a)\frac{1}{\theta} e^{-y/\theta} dy = a + \int_{y=0}^{y=\infty} y \frac{1}{\theta} e^{-y/\theta} dy = a+\theta$$ because y is just exponential.

$$\sigma^2 = \int_a^\infty (x-(\theta+a))^2 f(x)dx = \int_a^\infty (x-(\theta+a))^2 \frac{1}{\theta} e^{-(x-a)/\theta} dx$$ again let y=x-a

$$= \int_a^\infty (x-(\theta+a))^2 f(x)dx = \int_0^\infty (y-\theta)^2 \frac{1}{\theta} e^{-y/\theta} dy = \theta^2$$ because y is again just exponential.

b) Find the form of the MoM estimators for a and θ.

$$\bar{x} = \hat{a}_{mom} + \hat{\theta}_{mom}$$

$$\hat{\sigma}^2 = \hat{\theta}_{mom}{}^2 \Rightarrow \hat{\theta}_{mom} = \hat{\sigma} \Rightarrow \hat{a}_{mom} = \bar{x} - \hat{\sigma}$$

In this case we get $\hat{\theta}_{mom} = \hat{\sigma} = 3.242$ and $\hat{a}_{mom} = \bar{x} - \hat{\sigma} = 8.8 - 3.242 = 5.558$

c) Find the form of the MLE for a and θ. (Note that when you take the derivative with respect to a that it can never equal zero, so the maximum must happen at one of the end-points. Also note that the bigger a is the bigger the log-likelihood is. Based on what you know about the pdf, what is the biggest value that a can have?)

$$lik(a,\theta \mid x_1,...x_n) = f(x_1,...x_n \mid \theta) = \prod_{i=1}^n f(x_i \mid \theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-(x_i-a)/\theta}$$

$$\log lik(a,\theta \mid x_1,...x_n) = \sum_{i=1}^n [-\log\theta - \frac{(x_i-a)}{\theta}] = -n\log\theta - \sum_{i=1}^n \frac{x_i}{\theta} + \frac{na}{\theta}$$

$$\frac{\partial}{\partial a}\log lik(a,\theta \mid x_1,...x_n) = \frac{n}{\theta}$$ which is never 0! So must be a boundary. The biggest it can be is the smallest observed x!

$$\frac{\partial}{\partial \theta}\log lik(a,\theta \mid x_1,...x_n) = \frac{-n}{\theta} + \sum_{i=1}^n \frac{x_i}{\theta^2} - \frac{na}{\theta^2} = 0 \Rightarrow n\theta = \sum_{i=1}^n x_i - na \Rightarrow \theta = \bar{x} - a$$

So we get that $\hat{a}_{mle} = x_{(1)}$ and $\hat{\theta}_{mle} = \bar{x} - x_{(1)}$ which in this case are $\hat{a}_{mle} = 5.8$ and $\hat{\theta}_{mle} = 8.8 - 5.8 = 3$

3) Consider the Cauchy distribution centered at θ, that has pdf

$$f(x) = \frac{1}{\pi(1+(x-\theta)^2)} \text{ for } -\infty<x<\infty$$

a) Why can't there be a MoM estimator for θ?

The Cauchy distribution doesn't have any moments!

b) What formula must the MLE satisfy?

$$lik(\theta \mid x_1,\dots x_n) = f(x_1,\dots x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\pi(1+(x_i-\theta)^2)}$$

$$\log lik(\theta \mid x_1,\dots x_n) = \sum_{i=1}^{n} [-\log \pi - \log(1+(x_i-\theta)^2] = -n\log \pi - \sum_{i=1}^{n} \log(1+(x_i-\theta)^2)$$

$$\frac{\partial}{\partial \theta} \log lik(\theta \mid x_1,\dots x_n) = -\sum_{i=1}^{n} \frac{-2(x_i-\theta)}{1+(x_i-\theta)^2} = 0 \Rightarrow \hat{\theta}_{mle} \text{ must solve } \sum_{i=1}^{n} \frac{(x_i-\theta)}{1+(x_i-\theta)^2} = 0$$

c) Use R to find the estimate of θ based on the sample –0.6, 4.2, 1.1, -4.3, -10.3, 1.6, 4.8, 30.9, 0.4, 1.5.

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x<-c(-0.6, 4.2, 1.1, -4.3, -10.3, 1.6, 4.8, 30.9, 0.4, 1.5)

cauchynloglik<-function(theta,data){
   n<-length(data)
   x<-data
   -(-n*log(pi)-sum(log(1+(x-theta)^2)))}

optim(0,cauchynloglik,method="BFGS",data=x)

$par
[1] 1.171398
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