STAT 703/J703 – Spring 2005 - Take Home Exam 1

Due by Noon, Thursday, February 17th

Answer 10 of the 11 following questions (I will grade your best 10). Show all of your work for credit.
There are no "trick" questions, but some are decidedly easier than others.
Note that question 4 requires that you can answer both questions 2 and 3; and question 5 requires that you can
answer either question 2 or question 3.
You may not consult with anyone else on these problems; please contact me if you have any questions.

The Rayleigh distribution is used in reliability theory. It is a special case of the Weibull distribution with
shape=2 and scale=$\sqrt{2}\ \theta$. It has pdf, mean, and variance

$$f(x\mid\theta)=\frac{x\exp\left(-\dfrac{x^2}{2\theta^2}\right)}{\theta^2} \text{ for } x>0, \quad \mu=\theta\sqrt{\frac{\pi}{2}}, \quad \text{and } \sigma^2=\frac{4-\pi}{2}\theta^2$$

Problems 1-5 assume a random sample of size n from a Rayleigh distribution with parameter θ.

1) Find the formula for the method of moments estimator for θ.

2) Find the formula for the maximum likelihood estimator for θ.

3) Find the asymptotic variance of the mle for θ.   [Hint: $E(X^2)=2\ \theta^2$]

4) Construct an approximate 95% CI for the parameter θ for the following sample from a population with a
Rayleigh distribution.

2.3	3.1	3.4	4.0	4.1
4.5	4.8	6.0	6.1	9.7

5) Use the parametric bootstrap to estimate the standard error for one of the estimators (your choice of method
of moments or maximum likelihood) for a sample of size 20 from a Rayleigh distribution with $\theta=4$.   [Hint:
`rweibull(n, shape=2, scale=sqrt(2)*theta)`]

According to "Mathworld" the applications of the lognormal distribution include the size of silver particles in a
photographic emulsion, the survival time of bacteria in disinfectants, the weight and blood pressure of humans,
and the number of words written in sentences by George Bernard Shaw. The pdf, mean, and variance of the
lognormal are:

$$f(x\mid m,s)=\frac{1}{xs\sqrt{2\pi}}\exp\left[-\frac{(\ln(x)-m)^2}{2s^2}\right] \text{ for } x>0, \quad \mu=e^{\frac{1}{2}(2m+s^2)}, \quad \text{and } \sigma^2=e^{2m+2s^2}-e^{2m+s^2}$$

Problems 6-7 consider a random sample of size n from a lognormal distribution with parameters m and s.

6) Find the formula for the method of moments estimators for m and s.   [Hint: μ^2 has terms that will cancel
with σ^2 both by addition and division.]

7) Find the formula for the maximum likelihood estimators for m and s.

The F distribution has 2 parameters m and n which are usually taken to be degrees of freedom. They can actually have non-integers as well and the distribution has a mean and variance as long as m>0 and n>4. The pdf, mean, and variance of the F are:

$$f(x\,|\,m,n) = \frac{\Gamma\!\left(\dfrac{n+m}{2}\right)n^{\frac{n}{2}}m^{\frac{m}{2}}}{\Gamma\!\left(\dfrac{n}{2}\right)\Gamma\!\left(\dfrac{m}{2}\right)}\,\frac{x^{\frac{n}{2}-1}}{(m+nx)^{\frac{n+m}{2}}}\ \text{for } x>0,\ \ \mu = \frac{n}{n-2},\ \ \text{and } \sigma^2 = \frac{n^2(2m+2n-4)}{m(n-2)^2(n-4)}$$

Questions 8-10 consider a random sample of size 10 from an F distribution with parameters m and n.

| 0.13 | 0.18 | 0.31 | 0.34 | 0.44 |
| 1.01 | 1.16 | 2.77 | 3.46 | 5.87 |

8) The two parameters (m and n) could be estimated either by method of moments or by maximum likelihood. Name a statistical property that both of these estimates share. Then explain which of the two you would rather find the exact formula for if I asked you to for the exam. (Don't find it!)

9) Use R to find the maximum likelihood estimates for m and n based on the above sample using 5 and 6 as the initial estimates. [Hint: Notice you can write the log-likelihood in R as: `sum(log(df(x,m,n)))`]

10) Construct a q-q plot to check if the sample comes from an F distribution with parameters 5 and 6. Does it? If not, briefly describe why not in terms of how the data appears in comparison to the shape of the distribution. (e.g. the data is more skewed left, more skewed right, has heavier tails, etc…)

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11) Consider calculating the sample mean $\bar{x}$ for a random sample $x_1,\ldots x_n$ from a population X with mean μ and variance σ^2. What do we know about the sampling distribution of $\bar{x}$ if we know nothing about the distribution of X? What do we know about the sampling distribution of $\bar{x}$ if X is normally distributed?