STAT 702/J702

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-Lecture 18-

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Today

- Convolutions and Quotients (cont.)
- Order Statistics

3.6.1 - Special Case 1 – Convolution

In general say Z=X+Y

We can find a general formula for $F_Z(z)=P(Z<z)$ simply by finding the appropriate area under $f(x,y)$.

Taking the derivative then gives us the pdf.

Example) X and Y are exponential RVs with parameter λ.

<u>3.6.1 - Special Case 2 – Quotient</u>

A general formula for the quotient Z=Y/X can also be derived by examining the CDF.

To do this easily, note that if $y/x \leq z$ then if $x>0$ we have $y \leq xz$ and if $x<0$ then $y \geq xz$.

Back to the earlier example)

X and Y have joint p.d.f.

$f_{XY}(x,y) = 2 \quad 0 \leq x < y \leq 1$

U=X/Y

3.7 – Order Statistics

Let X_1, X_2, …. X_n be independent random variables with the same CDF $F_X(x)$.

The values in order from lowest to smallest are the order statistics $X_{(1)}$, $X_{(2)}$, …. $X_{(n)}$.

First consider the maximum $U=X_{(n)}$.

Note that $U \le u$ if and only if all of the $X_i \le u$.

$$F_U(u) = P(U \le u)$$
$$= P((X_1 \le u) \cap \cdots \cap (X_n \le u))$$
$$= P(X_1 \le u) \cdots P(X_n \le u)$$

$$= P(X_1 \le u) \cdots P(X_n \le u)$$
$$= F_X(u) \cdots F_X(u) = [F_X(u)]^n$$

$= P$

Taking the derivative we get:

$$f_U(u) = n f_X(u) [F_X(u)]^{n-1}$$

The minimum $V=X_{(1)}$ works similarly:

$$F_V(v) = 1 - [1 - F_X(v)]^n$$

$$f_V(v) = n f_X(v)[1 - F_X(v)]^{n-1}$$

Similar logic helps to get the marginal p.d.f. for any of the order statistics (as you will show in the homework).

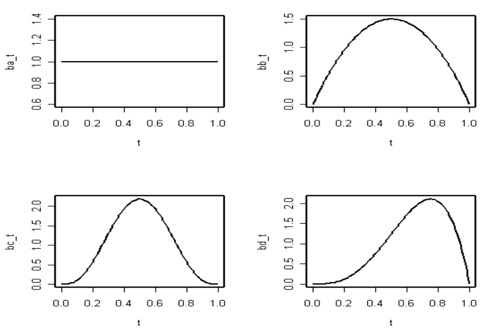
$$f_{X_{(k)}}(x_{(k)}) = \frac{n!}{(k-1)!(n-k!)} f_X(x_{(k)}) \cdot F^{k-1}(x_{(k)})[1 - F_X(x_{(k)})]^{n-k}$$

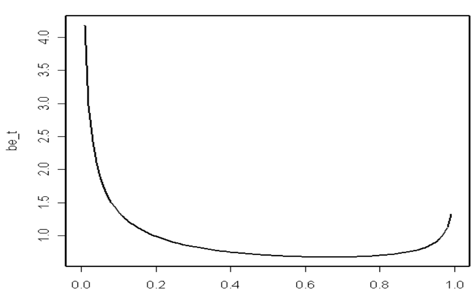
Example 1) Say you conduct 10 independent tests of hypotheses. How small should the smallest p-value be for you to reject it at a 0.05 level?

That is, what is the 5th-%ile for the 1st order statistic?

Example 2) $x_{(k)}$ for a uniform random variable.

This is a beta distribution with parameters k and n-k+1.





The joint p.d.f. of all of the order statistics is:

$$f_{X_{(1)},\ldots X_{(n)}}(x_{(1)},\ldots x_{(n)}) = n!\,f(x_{(1)})\cdots f(x_{(n)})$$

One way to find the joint p.d.f. of a pair of order statistics would be to integrate out the *n*-2 you are not concerned with.

Another way is to use what the text calls “a differential argument” (Theorem A on 101 uses this to prove the result in the hmwk.)

Say we want the joint p.d.f. of $X_{(i)}$ and $X_{(j)}$ where i<j.

The trick to getting the joint p.d.f. directly is try to let our insights into discrete distributions apply to continuous random variables.

In particular we will imagine that:

$$f(x,y) = P[x \le X \le x+dx, y \le Y \le y+dy]$$

$$f(x) = P[x \le X \le x+dx]$$

And so… $f_{X_{(i)},X_{(j)}}(x_{(i)},x_{(j)}) =$

$$\cdot\frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$
$$\cdot F_X{}^{i-1}(x_{(i)})$$
$$\cdot[F_X(x_{(j)})-F_X(x_{(i)})]^{j-i-1}$$
$$\cdot[1-F_X(x_{(j)})]^{n-j}$$
$$\cdot f_X(x_{(i)})f_X(x_{(j)})$$