Overview of the Two-way ANOVA

Factorial, With Replications, Balanced, Fixed Effect

Say there are two factors
Factor A has levels numbered i=1, ... A
Factor C has levels numbered j=1, ...C
and each combination has k=1, ... n replicatons.

*(Note that the names we give to the factors don't matter!
On page 343 of the text they use A and B...
on 348 they use G and R.)*

The data could be laid out as follows:

Factor A ▼	Factor C: j=1	j=2	...	j=C	Means for Factor A ▼
i=1	x_{111} x_{112} $\vdots$ x_{11n} $\bar{x}_{11}$	x_{121} x_{122} $\vdots$ x_{12n} $\bar{x}_{12}$	...	x_{1C1} x_{1C2} $\vdots$ x_{1Cn} $\bar{x}_{1C}$	$\bar{A}_1$
i=2	x_{211} x_{212} $\vdots$ x_{21n} $\bar{x}_{21}$	x_{221} x_{222} $\vdots$ x_{22n} $\bar{x}_{22}$	...	x_{2C1} x_{2C2} $\vdots$ x_{2Cn} $\bar{x}_{2C}$	$\bar{A}_2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
i=A	x_{A11} x_{A12} $\vdots$ x_{A1n} $\bar{x}_{A1}$	x_{A21} x_{A22} $\vdots$ x_{A2n} $\bar{x}_{A2}$	...	x_{AC1} x_{AC2} $\vdots$ x_{ACn} $\bar{x}_{AC}$	$\bar{A}_A$
Means for Factor C ▶	$\bar{C}_1$	$\bar{C}_2$	...	$\bar{C}_C$	$\bar{x}$

The model equation for this two way ANOVA could be written as:

$$x_{ijk} = \mu_{\text{baseline}} + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk} \quad \text{for } i=1,\ldots A,\ j=1,\ldots C, \text{ and } k=1,\ldots n$$

where the x_{ijk} are the observations

μ_{baseline} is the baseline

$\alpha_1, \alpha_2, \ldots \alpha_A$ are the main effects for the levels of factor A

$\gamma_1, \gamma_2, \ldots \gamma_C$ are the main effects for the levels of factor C

$(\alpha\gamma)_{11}, (\alpha\gamma)_{12}, \ldots (\alpha\gamma)_{21}, \ldots (\alpha\gamma)_{AC}$ are the interactions for the combinations of A and C

and the ε_{ijk} are the errors that satisfy the conditions of mean equal to 0, equal variances, normality, and independence

The basic ANOVA table could then be written as:

Source	SS	df	MS	F
Between	$SS_{bet} = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(\bar{x}_{ij} - \bar{x})^2$	$AC-1$	$MS_{bet} = \frac{SS_{bet}}{AC-1}$	$F = \frac{MS_{bet}}{MS_{wit}}$
▶Factor A	$SS_A = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(\bar{A}_i - \bar{x})^2$	$A-1$	$MS_A = \frac{SS_A}{A-1}$	$F = \frac{MS_A}{MS_{wit}}$
▶Factor C	$SS_C = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(\bar{C}_j - \bar{x})^2$	$C-1$	$MS_C = \frac{SS_C}{C-1}$	$F = \frac{MS_C}{MS_{wit}}$
▶AC Interaction	$SS_{AC} = SS_{bet} - SS_A - SS_C$	$(AC-1)-(A-1)-(C-1)$ $=(A-1)(C-1)$	$MS_{AC} = \frac{SS_{AC}}{(A-1)(C-1)}$	$F = \frac{MS_{AC}}{MS_{wit}}$
Within	$SS_{wit} = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(x_{ijk} - \bar{x}_{ij})^2$	$ACn-AC$	$MS_{bet} = \frac{SS_{bet}}{ACn-AC}$	
Total	$SS_{tot} = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(x_{ijk} - \bar{x})^2$	$ACn-1$		

The formulas in this ANOVA table can be simplified to the following:

Source	SS	df	MS	F
Between	$SS_{bet} = n\sum_{i=1}^{A}\sum_{j=1}^{C}(\bar{x}_{ij} - \bar{x})^2$	*AC-1*	$MS_{bet} = \frac{SS_{bet}}{AC-1}$	$F = \frac{MS_{bet}}{MS_{wit}}$
►Factor A	$SS_A = nC\sum_{i=1}^{A}(\bar{A}_i - \bar{x})^2$	*A-1*	$MS_A = \frac{SS_A}{A-1}$	$F = \frac{MS_A}{MS_{wit}}$
►Factor C	$SS_C = nA\sum_{j=1}^{C}(\bar{C}_j - \bar{x})^2$	*C-1*	$MS_C = \frac{SS_C}{C-1}$	$F = \frac{MS_C}{MS_{wit}}$
►AC Interaction	$SS_{AC} = n\sum_{i=1}^{A}\sum_{j=1}^{C}(\bar{x}_{ij} - \bar{A}_i - \bar{C}_j + \bar{x})^2$	*AC-A-C+1* *=(A-1)(C-1)*	$MS_{AC} = \frac{SS_{AC}}{(A-1)(C-1)}$	$F = \frac{MS_{AC}}{MS_{wit}}$
Within	$SS_{wit} = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(x_{ijk} - \bar{x}_{ij})^2$	*AC(n-1)*	$MS_{bet} = \frac{SS_{bet}}{AC(n-1)}$	
Total	$SS_{tot} = \sum_{i=1}^{A}\sum_{j=1}^{C}\sum_{k=1}^{n}(x_{ijk} - \bar{x})^2$	*ACn-1*		

Notice that there are four F statistics (each of which has its own p-value).
The null hypotheses tested by these F statistics are:

Between (Omnibus Test) H_0: $\alpha_1 = \ldots = \alpha_A$ and $\gamma_1 = \ldots = \gamma_C$ and $(\alpha\gamma)_{11} = (\alpha\gamma)_{12} = \ldots = (\alpha\gamma)_{21} = \ldots (\alpha\gamma)_{AC}$

Factor A (Factor A has no effect) H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_A$

Factor C (Factor C has no effect) H_0: $\gamma_1 = \gamma_2 = \ldots = \gamma_C$

Interaction AC (No Interactions) H_0: $(\alpha\gamma)_{11} = (\alpha\gamma)_{12} = \ldots = (\alpha\gamma)_{21} = \ldots (\alpha\gamma)_{AC}$

The reason that these various F-tests test the hypotheses we want can be seen by looking at the expected values of the mean squares. If we write the model so that the baseline sets the α, γ, and interactions to have mean zero, then the expected mean squares can be written as:

$$E(MS_{bet}) = \sigma_\varepsilon^2 + \frac{Cn}{AC-1}\sum_{i=1}^{A}\alpha_i^2 + \frac{An}{AC-1}\sum_{j=1}^{C}\gamma_i^2 + \frac{n}{AC-1}\sum_{i=1}^{A}\sum_{j=1}^{C}(\alpha\gamma)_{ij}^2$$

$$E(MS_A) = \sigma_\varepsilon^2 + \frac{Cn}{A-1}\sum_{i=1}^{A}\alpha_i^2$$

$$E(MS_C) = \sigma_\varepsilon^2 + \frac{An}{C-1}\sum_{j=1}^{C}\gamma_i^2$$

$$E(MS_{AC}) = \sigma_\varepsilon^2 + \frac{n}{(A-1)(C-1)}\sum_{i=1}^{A}\sum_{j=1}^{C}(\alpha\gamma)_{ij}^2$$

$$E(MS_{wit}) = \sigma_\varepsilon^2$$

So, to test that there is no effect due to factor A, we would need to cancel out the σ_ε^2 in the $E(MS_A)$. We could do this by dividing the MS_A by the MS_{wit}, which is exactly what happens in the ANOVA table. The other tests are made similarly.

Note 1: The basic rules (how to make the MS and F) for higher-way ANOVA tables are the same as for the One-way ANOVA. The only major difference is that we are splitting up the SS_{bet}.

Note 2: The construction of a three-way ANOVA table works similarly to that of the above two-way table. There are a variety of books on ANOVA and design of experiments that give the formulas if you need them.

Note 3: The formulas in the ANOVA table above, and the argument about the F tests, only work if the design is Factorial, With Replications, Balanced, and Fixed Effect