

# Random Effects Models

## Supplement for Chapter 8

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The model equation for a two-way balanced, factorial design, with replications can be written:

$y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk}$  for  $i=1, \dots, A$ ,  $j=1, \dots, C$ , and  $k=1, \dots, n$ . The  $\alpha$  are the factor 1 levels, the  $\gamma$  are the factor 2 levels, the  $\alpha\gamma$  are the interactions, and the  $\varepsilon$  are the errors. The main  $F$ -test in the ANOVA table tests

$$\begin{aligned} H_0: & \alpha_1 = \dots = \alpha_A \text{ and } \gamma_1 = \dots = \gamma_C \text{ and } (\alpha\gamma)_{11} = \dots = (\alpha\gamma)_{AC} \\ \text{vs. } & H_A: \text{ at least one is not equal} \end{aligned}$$

While the Type III (and Type I) sum of squares can be used to test the three parts of the above hypothesis separately.

The reason that these various  $F$ -tests test the hypotheses we want can be seen by looking at the expected values of the mean squares. If we write the model so that the  $\alpha$ ,  $\gamma$ , and interactions have mean zero, then the expected mean squares can be written as:

$$E(\text{MS}_{\text{bet}}) = \sigma_{\varepsilon}^2 + \frac{Cn}{AC-1} \sum_{i=1}^A \alpha_i^2 + \frac{An}{AC-1} \sum_{j=1}^C \gamma_j^2 + \frac{n}{AC-1} \sum_{i=1}^A \sum_{j=1}^C (\alpha\gamma)_{ij}^2$$

$$E(\text{MS}_A) = \sigma_{\varepsilon}^2 + \frac{Cn}{A-1} \sum_{i=1}^A \alpha_i^2$$

$$E(\text{MS}_C) = \sigma_{\varepsilon}^2 + \frac{An}{C-1} \sum_{j=1}^C \gamma_j^2$$

$$E(\text{MS}_{AC}) = \sigma_{\varepsilon}^2 + \frac{n}{(A-1)(C-1)} \sum_{i=1}^A \sum_{j=1}^C (\alpha\gamma)_{ij}^2$$

$$E(\text{MS}_{\text{res}}) = \sigma_{\varepsilon}^2$$

In order to test  $\alpha_1 = \dots = \alpha_A$  we would normally find the  $F$ -statistic by dividing the  $\text{MS}_A$  by the  $\text{MS}_{\text{res}}$ . In the case where the null hypothesis is true, this would be dividing something with expected value  $\sigma_{\varepsilon}^2$ , by something with expected value  $\sigma_{\varepsilon}^2$  (note in the  $E(\text{MS}_A)$  that all the  $\alpha_i$  are 0). We would thus expect the  $F$ -value to be near 1, and we would fail to reject  $H_0$ . In the case where not all the  $\alpha_i$  are zero, the numerator would be  $\sigma_{\varepsilon}^2$  plus at least one positive term (based on the  $\alpha_i$  squared). We would thus expect the  $F$ -value to be greater than 1, and would have a chance to reject  $H_0$ . This works the same way for testing if factor 2 is significant, or if there are any significant interactions. For the overall  $F$ -value from the ANOVA table, notice that the only way the numerator can have an expected value of  $\sigma_{\varepsilon}^2$  is if all of factor 1, factor 2, and the interactions are zero. Usually we don't worry about these expected mean squares, since we know everything will work out ok. The subject of this supplement, random effects, complicates matters though, and we will have to start paying attention to the expected mean squares.

The basic ANOVA problems we have dealt with so far are referred to as *fixed effects models*. We are interested in these particular levels of the factors, and so test whether they are all equal to each other or not. Even if we accept that the effects of  $\alpha_1, \alpha_2, \dots, \alpha_A$  are all equal, we can make no conclusion about the other possible  $\alpha$  values we chose not to include in our experiment.

We can imagine a different situation however. Say that the response  $y$  is the time it takes for rats to complete a maze. We imagine that gender might have an effect, and so we set  $\gamma_1$ =male and  $\gamma_2$ =female. Our primary topic of interest though is the effect that different brands of rat feed have on their times. We don't have enough time or money to try all of the thousands of different brands of rat feed, so we will take a random sample of five brands. The hypothesis we would like to test is no longer  $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5$ . Instead, we want to know whether the entire population of brands of rat feed all have the same effect. One way of writing this is to say we want to test whether  $\sigma_\alpha^2=0$ , because if the variance of a population is zero, then they all have the same value. This sampling of the  $\alpha$ s we examine from all of the possible  $\alpha$ s is what produces a *random effects model*.

Because of the extra randomness in random effects models, the expected mean squares change. For a one-way ANOVA, they would change from

$$E(\text{MS}_{\text{bet}}) = \sigma_\epsilon^2 + \frac{n}{A-1} \sum_{i=1}^A \alpha_i^2 \quad \text{and} \quad E(\text{MS}_{\text{wit}}) = \sigma_\epsilon^2$$

for the fixed effect model to

$$E(\text{MS}_{\text{bet}}) = \sigma_\epsilon^2 + n\sigma_\alpha^2 \quad E(\text{MS}_{\text{wit}}) = \sigma_\epsilon^2$$

for the random effects model. Notice that  $F = \text{MS}_{\text{bet}} / \text{MS}_{\text{wit}}$  tests that the specific  $\alpha_i$  are equal for the fixed effect model, but tests that  $\sigma_\alpha^2=0$  for the random effects model! It thus looks like the hypothesis doesn't make any difference to which  $F$  statistic we use. However, if we go to a two-way model things get more complicated.

In a two-way ANOVA, we have to decide which effects we want to be random. We could have both factors and the interactions be random effects. We could also have just one of the factors be random. In this case, called a *mixed effects model*, we then have to decide if we want the interactions to be random or not. How exactly to treat the interactions is a rather complicated question, and even has options that could be described as being in between random and fixed! In any case, if you have a two (or more)-way ANOVA with any random effects, you would be wise to talk to a statistical consultant to make sure you have chosen the correct options. If you want to do more reading on your own, additional material can be found in: George W. Cobb, *Introduction to Design and Analysis of Experiments*, New York: Springer, 1998, pages 554-608, or John Neter, et.al., *Applied Linear Statistical Models*, 4<sup>th</sup> Edition, Boston: McGraw-Hill, 1996, Chapter 24.

In some cases (but not all!) once you have decided on a model, you can use the SAS output to determine what MS you need to use to construct the tests of hypotheses.

The SAS code below analyzes the data in Table 8-11 (page 365), taking the strain of rat to be a random effect. That is, we are told that the two strains of rat were chosen from a large number of possible strains, and just happen to be called “Normal” and “Hypertensive”.

```

DATA ksh;
INPUT strain $ site $ activity @@;
CARDS;
norm dct 62      norm dct 73      norm dct 58      norm dct 66
hyp  dct 44      hyp  dct 49      hyp  dct 46      hyp  dct 37
norm ccd 15      norm ccd 31      norm ccd 19      norm ccd 35
hyp  ccd 8       hyp  ccd 36      hyp  ccd 11      hyp  ccd 18
norm omcd 7      norm omcd 7      norm omcd 9      norm omcd 17
hyp  omcd 19     hyp  omcd 7      hyp  omcd 15     hyp  omcd 4
;

PROC GLM DATA=ksh ORDER=DATA;
CLASS strain site;
MODEL activity = strain site strain*site;
RANDOM strain;
RUN;

```

The line RANDOM contains a list of all of the effects (or interactions) that we want to treat as random. In this case we have only put strain there. The bottom of the SAS output then contains the formulas for the expected mean squares.

The GLM Procedure	
Source	Type III Expected Mean Square
strain	Var(Error) + 12 Var(strain) + Q(strain*site)
site	Var(Error) + Q(site, strain*site)
strain*site	Var(Error) + Q(strain*site)

In the SAS notation,  $\text{Var}(\text{\_factor name\_})$  means simply the variance of that factor. For balanced designs,  $Q(\text{\_factor name\_})$  means some constant times the sum of those factor effects squared. If we use  $\text{strain}_i$  to represent the effect of strain  $i$ , and  $\text{site}_j$  to represent the effect of site  $j$ , then the above output then says that

$$E(\text{MS}_{\text{strain}}) = \sigma_{\mathcal{E}}^2 + 12\sigma_{\text{strain}}^2 + (\text{constant}) \sum_{i=1}^2 \sum_{j=1}^3 (\text{strain} * \text{site})_{ij}^2$$

$$E(\text{MS}_{\text{strain*site}}) = \sigma_{\mathcal{E}}^2 + (\text{constant}) \sum_{i=1}^2 \sum_{j=1}^3 (\text{strain} * \text{site})_{ij}^2$$

and, as always, we have

$$E(\text{MS}_{\text{res}}) = \sigma_{\mathcal{E}}^2$$

For  $E(\text{MS}_{\text{site}})$  we don't know from the output what constants multiply the sum of  $\text{site}_j^2$  and  $(\text{strain} * \text{site})_{ij}^2$ ; thus we can't be certain that we would use them correctly. Because the other two  $E(\text{MS})$  only involve  $\text{strain} * \text{site}$ , and because the design is balanced, we know that those constants are the same.

We could thus test  $H_0: \sigma_{strain}^2 = 0$  by forming  $F = MS_{strain} / MS_{strain*site}$ . This is because the two MS will have the same expected value if the null hypothesis is true. The degrees of freedom would be the ones corresponding to the two MS that were used.

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	9242.37500	1848.47500	28.73	<.0001
Error	18	1158.25000	64.34722		
Corrected Total	23	10400.62500			

  

Source	DF	Type III SS	Mean Square	F Value	Pr > F
strain	1	459.375000	459.375000	7.14	0.0156
site	2	8287.000000	4143.500000	64.39	<.0001
strain*site	2	496.000000	248.000000	3.85	0.0404

From the output, this would give  $F = 459.375 / 248.000 = 1.8523$  and degrees of freedom 1,2. From table E-2, we see that this is much less than the value of 98.49 for  $\alpha = 0.05$  and we fail to reject  $H_0$ .

This leads naturally to the question of what the p-value of 0.0156 (the one for strain in the Type III tests) is testing. The Type III tests are always formed by taking the MS for that “source” and dividing it by the  $MS_{res}$ ; in this case  $7.14 = 459.375 / 64.34722$ . The degrees of freedom are the ones that correspond to the df; in this case 1 and 18. If we look at the E(MS) for these however, we find that this p-value is actually testing that both the variance of the strain effects is zero and that all of the interactions terms are equal. (Recall that we can tell this by simply comparing how the two expected mean squares differ. In this case the quotient of the two will equal one only if the variance of the strain effects is zero, and all of the terms from the interactions are zero.)

You are certainly not expected to become experts in random and fixed effects models, because they can become very complicated very quickly. You should however be able to recognize when one of the main effects in your model will be random. (If so, you will need to confer with a consultant or a reference to see what to do with the interactions.) For balanced designs, like the above example, you should also be able to use the expected mean squares (that you are given) to test a particular hypothesis and to tell what hypotheses the Type III tests that SAS gives are testing.