

- 16.10 Let Y_1, Y_2, \dots, Y_n denote a random sample from an exponentially distributed population with density $f(y|\theta) = \theta e^{-\theta y}, 0 < y$. (Note: the mean of this population is $\mu = 1/\theta$.) Use the conjugate gamma (α, β) prior for θ to do the following.

a Show that the joint density of $Y_1, Y_2, \dots, Y_n, \theta$ is

$$f(y_1, y_2, \dots, y_n, \theta) = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp \left[-\theta \left(\frac{\beta}{\beta \sum y_i + 1} \right) \right].$$

b Show that the marginal density of Y_1, Y_2, \dots, Y_n is

$$m(y_1, y_2, \dots, y_n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta}{\beta \sum y_i + 1} \right)^{\alpha+n}.$$

c Show that the posterior density for $\theta | (y_1, y_2, \dots, y_n)$ is a gamma density with parameters $\alpha^* = n + \alpha$ and $\beta^* = \beta / (\beta \sum y_i + 1)$.

d Show that the Bayes estimator for $\mu = 1/\theta$ is

$$\hat{\mu}_B = \frac{\sum Y_i}{n + \alpha - 1} + \frac{1}{\beta(n + \alpha - 1)}.$$

[Hint: Recall Exercise 4.111(e).]

e Show that the Bayes estimator in part (d) can be written as a weighted average of \bar{Y} and the prior mean for $1/\theta$. [Hint: Recall Exercise 4.111(e).]

- 4.111 Suppose that Y has a gamma distribution with parameters α and β .

a If a is any positive or negative value such that $\alpha + a > 0$, show that

$$E(Y^a) = \frac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}.$$

b Why did your answer in part (a) require that $\alpha + a > 0$?

c Show that, with $a = 1$, the result in part (a) gives $E(Y) = \alpha/\beta$.

d Use the result in part (a) to give an expression for $E(\sqrt{Y})$. What do you need to assume about α ?

e Use the result in part (a) to give an expression for $E(1/Y)$, $E(1/\sqrt{Y})$, and $E(1/Y^2)$. What do you need to assume about α in each case?

4.111 d $\sqrt{\beta} \Gamma\left(\alpha + \frac{1}{2}\right) / \Gamma(\alpha)$ if $\alpha > 0$

e $\frac{1}{\beta(\alpha-1)}$ if $\alpha > 1$, $\frac{\Gamma(\alpha - \frac{1}{2})}{\sqrt{\beta} \Gamma(\alpha)}$
if $\alpha > \frac{1}{2}$, $\frac{1}{\beta^2(\alpha-1)(\alpha-2)}$
if $\alpha > 2$