Midterm Exam

1. Consider the "trapezoidal" density function:

$$g(x) = -1.25x + 2.875,\ 1 \leq x \leq 2.$$

   - Show how to generate random deviates from this distribution using the probability integral transform.
   - Show how to generate random deviates from this distribution using a mixture of a uniform distribution and a triangular distribution.
   - Discuss the advantages/disadvantages of the above methods.
   - Implement both methods by generating 500 deviates from $G(\cdot)$; evaluate your results.

2. We had earlier considered several different strategies for evaluating

$$\int_1^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

   .

   Monte Carlo integration from a $Unif(1,2)$ distribution was more efficient than either Monte Carlo integration or importance sampling based on a $N(0,1)$ distribution. Consider an importance sampling approach based on the trapezoidal distribution from Problem 1 with $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, $f(x)$ a $Unif(1,2)$ density and $g(x)$ as defined above.

   - Based on results from our notes and the text, explain why this approach could be expected to be efficient.
   - Compute the variance of the theoretical lower bound for an importance sampling estimator, the variance of the proposed importance sampling estimator and the variance of the usual $Unif(1,2)$ Monte Carlo estimator. Use analytical methods when possible. Comment.
   - Confirm your theoretical results with a simulation exercise. You should assume samples of size 10 and 25 and use 1000 simulations.

3. Consider optimizing the likelihood for a random sample of $N(\mu, \sigma^2)$ random variables.

   - Write the negative of the log likelihood as a function of μ and ω, where $\omega = \sigma^2$. Find the gradient and Hessian for this function.
   - Write a script or function in **R** to compute the Newton-Raphson "step" for a single iterate–do NOT compute the inverse of the Hessian to do this.
   - Generate 20 standard normal random deviates; calculate the MLE of (μ, ω). Using a start value of $(0,1)$, calculate the first NR step for your function. Compare to the MLE and comment. Take a few more steps until you apparently have convergence (I know this goes against what I taught you in class, but compare your output to the calculated MLEs till they are equivalent to 6 significant digits). How many steps did your algorithm use?
   - Use `nlm` in **R** to find the MLE's compare the output from *nlm* to your results.