Three-parameter Gamma MLE

The three-parameter Gamma distribution, also referred to as a Pearson Type III distribution, is often usd in hydrology to model the log of flow data for river systems. The likelihood function for this distribution can be written as:

$$L(\alpha, \beta, \gamma|\mathbf{y}) = \sum_{i=1}^{n} \left[(\alpha - 1)\ln(y_i - \gamma) - \frac{y_i - \gamma}{\beta} - \alpha \ln \beta - \ln \Gamma(\alpha) \right],$$

where α is the shape parameter, β is the scale parameter, and γ is the shift or location parameter. The gradient for this log likelihood function has components:

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \ln(y_i - \gamma) - n\ln(\beta) - n\psi(\alpha)$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \frac{y_i - \gamma}{\beta^2} - \frac{n\alpha}{\beta}$$

$$\frac{\partial L}{\partial \gamma} = -\sum_{i=1}^{n} \frac{\alpha - 1}{y_i - \gamma} + \frac{n}{\beta},$$

where $\psi(\cdot)$ is the "psi" function, i.e., the derivative of $\ln(\Gamma(\alpha))$ with respect to α.

If we set the gradient to $\mathbf{0}$, the resulting likelihood equations can be intractable. Note that for $\hat{\alpha}$ less than 1, some of the y_i must be less than $\hat{\gamma}$, a logical inconsistency. In this case, the MLE does not exist. Many alternate approaches have been suggested to resolve this problem with the MLE, but we will proceed as though a solution exists.

From the gradient, we can compute the Hessian:

$$H(\alpha, \beta, \gamma) = \begin{bmatrix} -n\psi'(\alpha) & \frac{-n}{\beta} & -\sum_{i=1}^{n} \frac{1}{y_i - \gamma} \\ \frac{-n}{\beta} & -\sum_{i=1}^{n} \frac{y_i - \gamma}{\beta^3} + \frac{n\alpha}{\beta^2} & \frac{-n}{\beta^2} \\ -\sum_{i=1}^{n} \frac{1}{y_i - \gamma} & \frac{-n}{\beta^2} & -\sum_{i=1}^{n} \frac{\alpha - 1}{(y_i - \gamma)^2} \end{bmatrix}$$

If you work through the math, the information matrix actually has the rather straightforward form:

$$I(\alpha, \beta, \gamma) = n \begin{bmatrix} \psi'(\alpha) & \frac{1}{\beta} & \frac{1}{(\alpha-1)\beta} \\ \frac{1}{\beta} & \frac{\alpha}{\beta^2} & \frac{1}{\beta^2} \\ \frac{1}{(\alpha-1)\beta} & \frac{1}{\beta^2} & \frac{1}{(\alpha-1)\beta^2} \end{bmatrix}$$

This information could then be supplied to **R** to find MLE's and their standard errors.