2.3.4 Kendall coordinates: Planar case

Kendall coordinates are similar to Bookstein coordinates but location is removed in a different manner. We first need to define the Helmert sub-matrix which is used to remove location.

The Helmert sub-matrix H is the $(k-1) \times k$ Helmert matrix without the first row. The full Helmert matrix H^F, which is commonly used in Statistics, is a square $k \times k$ orthogonal matrix with its first row of elements equal to $1/\sqrt{k}$, and the remaining rows are orthogonal to the first row. We drop the first row of H^F so that the transformed HX does not depend on the original location of the configuration.

Definition 2.5 *The* j*th row of the* **Helmert sub-matrix** H

is given by

$$(h_j, ..., h_j, -jh_j, 0, ..., 0)\ , \quad h_j = -\{j(j+1)\}^{-1/2}, \tag{2.9}$$

and so the jth row consists of h_j repeated j times, followed by $-jh_j$ and then $k-j-1$ zeros, $j = 1, ..., k-1$.

For $k = 3$ the full Helmert matrix is explicitly

$$H^F = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

and the Helmert sub-matrix is

$$H = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$

For $k = 4$ points the full Helmert matrix is

$$H^F = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & 3/\sqrt{12} \end{bmatrix}$$

and the Helmert sub-matrix is

$$H = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & 3/\sqrt{12} \end{bmatrix}.$$

Consider the original complex landmarks $z^o = (z_1^o, ..., z_k^o)^{\mathrm{T}}$ and remove location by pre-multiplying by the Helmert sub-matrix H to give $z_H = Hz^o = (z_1, ..., z_{k-1})^{\mathrm{T}}$.

Definition 2.6 *The* **Kendall coordinates** *are given by*

$$u_j^K + iv_j^K = \frac{z_{j-1}}{z_1}, \quad j = 3, \ldots, k. \tag{2.10}$$

There is a simple 1-1 correspondence between Kendall and Bookstein coordinates. If we write

$$w^B = (u_3^B + iv_3^B, ..., u_k^B + iv_k^B)^{\mathrm{T}}$$

for Bookstein coordinates and

$$w^K = (u_3^K + iv_3^K, ..., u_k^K + iv_k^K)^{\mathrm{T}}$$

for Kendall coordinates, then it follows that

$$w^K = \sqrt{2} H_1 w^B \qquad (2.11)$$

where H_1 is the lower right $(k-2)\times(k-2)$ partition matrix of the Helmert sub-matrix H. Note that

$$H_1^{\mathrm{T}} H_1 = I_{k-2} - \frac{1}{k} 1_{k-2} 1_{k-2}^{\mathrm{T}} \ , \quad |H_1|^2 = 2/k, \quad (H_1^{\mathrm{T}} H_1)^{-1} = I_{k-2} + \frac{1}{2} 1_{k-2} 1_{k-2}^{\mathrm{T}}$$

so linear transformation from one coordinate system to the other is straightforward. The inverse transformation is

$$w^B = (H_1^{\mathrm{T}} H_1)^{-1} H_1^{\mathrm{T}} w^K / \sqrt{2}.$$

For $k = 3$ we have the relationship

$$u_3^B + iv_3^B = \frac{z_3^o - \frac{1}{2}(z_1^o + z_2^o)}{z_2^o - z_1^o} = \frac{\sqrt{3}}{2}(u_3^K + iv_3^K)$$

and so Kendall coordinates in this case are the coordinates of the third landmark after transforming landmarks 1 and 2 to $(-1/\sqrt{3}, 0)$ and $(1/\sqrt{3}, 0)$ by the similarity transformations. The total transformation from (z_1^o, z_2^o, z_3^o) to Kendall coordinates is

$$z_1^o \to -\frac{1}{\sqrt{3}}\ ,\ z_2^o \to \frac{1}{\sqrt{3}}\ ,\ z_3^o \to u_3^K + iv_3^K .$$

Throughout the text we shall often refer to the real $(2k - 4)$-vector of Kendall coordinates $u^K = (u_3^K, \ldots, u_k^K, v_3^K, \ldots, v_k^K)^{\mathrm{T}}$, stacking the coordinates in this particular order.

2.3.5 Kendall's spherical coordinates for triangles

For $k = 3$ we will see in Section 4.2.4 that the shape space is a sphere with radius $\frac{1}{2}$. A mapping from Kendall coordinates to the sphere of radius $\frac{1}{2}$ is

$$x = \frac{1 - r^2}{2(1 + r^2)}\ \ ,\ \ y = \frac{u_3^K}{1 + r^2}\ \ ,\ \ z = \frac{v_3^K}{1 + r^2} \qquad (2.12)$$

and $r^2 = (u_3^K)^2 + (v_3^K)^2$, so that $x^2 + y^2 + z^2 = \frac{1}{4}$, where u_3^K and v_3^K are Kendall coordinates of Section 2.3.4.

Definition 2.7 Kendall's spherical coordinates (θ, ϕ) *are given by the polar coordinates*

$$\tfrac{1}{2}\sin\theta\cos\phi = \frac{1-r^2}{2(1+r^2)}\ ,\ \tfrac{1}{2}\sin\theta\sin\phi = \frac{u_3^K}{1+r^2}\ \ ,\ \ \tfrac{1}{2}\cos\theta = \frac{v_3^K}{1+r^2}, \tag{2.13}$$

where $0 \le \theta \le \pi$ *is the angle of latitude and* $0 \le \phi < 2\pi$ *is the angle of longitude.*

The relationship between (u_3^K, v_3^K) and the spherical shape variables (Mardia, 1989c) is given by

$$\begin{aligned} u_3^K &= \frac{\sin\theta\sin\phi}{1+\sin\theta\cos\phi}, \\ v_3^K &= \frac{\cos\theta}{1+\sin\theta\cos\phi}. \end{aligned} \tag{2.14}$$

The sphere can be partitioned into 6 lunes and 12 half-lunes. In order to make the terminology clear, one example full-lune is $0 \leq \phi \leq \pi/3, 0 \leq \theta \leq \pi$ and one example half-lune is $0 \leq \phi \leq \pi/3, 0 \leq \theta \leq \pi/2$.

In Figure 27 we see triangle shapes located on the spherical shape space. The equilateral triangle with anti-clockwise labelling corresponds to the 'North pole' ($\theta = 0$) and the reflected equilateral triangle (with clockwise labelling) is at the 'South pole' ($\theta = \pi$). The flat triangles (three collinear points) lie around the equator ($\theta = \pi/2$). The isosceles triangles lie on the meridians $\phi = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$. The right-angled triangles lie on three small circles given by

$$\sin\theta\cos\left(\phi - \frac{2k\pi}{3}\right) = \tfrac{1}{2}\ ,\ k = 0, 1, 2,$$

and we see the arc of unlabelled right-angled triangles on the front half-lune in Figure 27.

Reflections of triangles in the upper hemisphere at (θ, ϕ) are located in the lower hemisphere at $(\pi - \theta, \phi)$. In addition, permuting the triangle labels gives rise to points in each of the six equal half-lunes in each hemisphere. Thus, if invariance under labelling and reflection was required, then we would be restricted to one of these half-lunes, for example the sphere surface defined by $0 \leq \phi \leq \pi/3, 0 \leq \theta \leq \pi/2$. Consider a triangle with labels A,B and C, and edge lengths AB, BC and AC. If the labelling and reflection of the points was unimportant, then we could relabel each triangle so that, for example, AB $\geq$ AC $\geq$ BC and point C is above the baseline AB.

For practical analysis and the presentation of data it is often desirable to use a suitable projection of the sphere for triangle shapes. Kendall (1983) defined an equal area

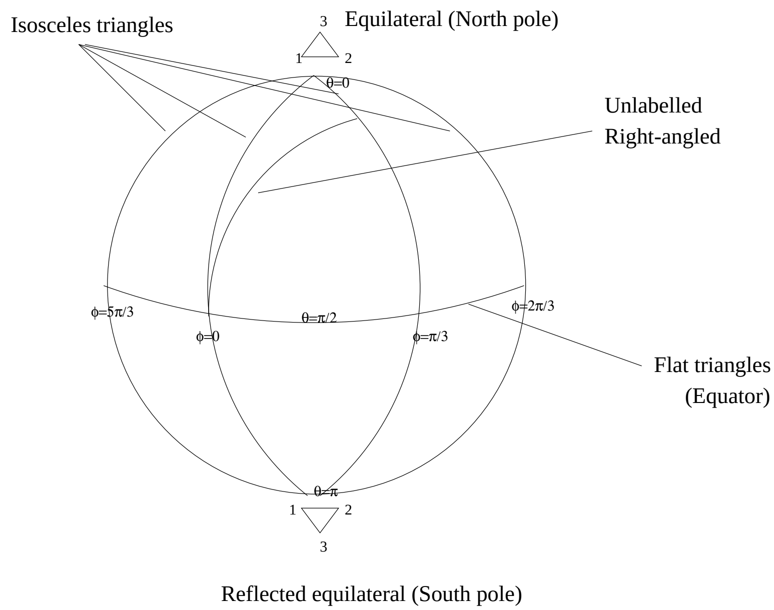

Figure 27 Kendall's spherical shape space for triangles in $m = 2$ dimensions. The shape coordinates are the latitude θ (with zero at the North pole) and the longitude ϕ.

projection of one of the half-lunes of the shape sphere to display unlabelled triangle shapes. The projected lune is bell-shaped and this graphical tool is also known as ‘Kendall’s Bell’ or the spherical blackboard (an example is given later in Figure 136).

An alternative equal-area projection is the Schmidt net (Mardia, 1989c) otherwise known as the Lambert projection given by

$$\xi = 2\sin\left(\frac{\theta}{2}\right)\ , \quad \psi = \phi\ ;\ 0 \le \xi \le \sqrt{2}, 0 \le \psi < 2\pi.$$

In Figure 28 we see a plot of one of the half-lunes on the upper hemisphere of shape space projected onto the Schmidt net. Example triangles are drawn with their centroids at polar coordinates (ξ, ψ) in the Schmidt net.

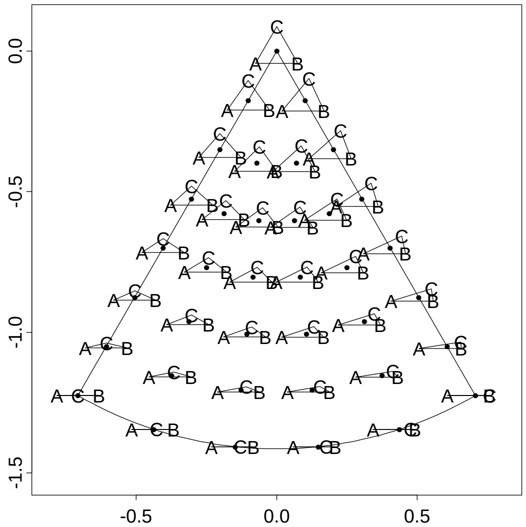

Figure 28 Part of the shape space of triangles projected onto the equal-area projection Schmidt net. If relabelling and reflection was not important, then all triangles could be projected into this sector.