Q6. Let $A = \sum_{j=1}^{m}\sum_{i=1}^{k}\sum_{l=1}^{k}\left(x_{ij} - x_{lj}\right)^2$

$$= \sum_{j=1}^{m}\sum_{i=1}^{k}\sum_{l=1}^{k}\left(x_{ij} - \bar{x}_j + \bar{x}_j - x_{lj}\right)^2$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{k}\sum_{l=1}^{k}\left\{\left(x_{ij} - \bar{x}_j\right)^2 + \left(\bar{x}_j - x_{lj}\right)^2 + 2\left(x_{ij} - \bar{x}_j\right)\left(\bar{x}_j - x_{lj}\right)\right\}$$

$$= k\sum_{j=1}^{m}\sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right)^2 + k\sum_{j=1}^{m}\sum_{l=1}^{k}\left(\bar{x}_j - x_{lj}\right)^2 + 2\sum_{j=1}^{m}\sum_{l=1}^{k}\left(\bar{x}_j - x_{lj}\right)\sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right)$$

$$= k\sum_{j=1}^{m}\sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right)^2 + \underbrace{k\sum_{j=1}^{m}\sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right)^2}_{\text{change subscript } l \text{ to } i.} + 2\sum_{j=1}^{m}\sum_{l=1}^{k}\left(\bar{x}_j - x_{lj}\right)\underbrace{\left(\sum_{i=1}^{k} x_{ij} - \frac{1}{k}k\cdot\sum_{i=1}^{k} x_{ij}\right)}_{=0}$$

$$= 2k\sum_{j=1}^{m}\sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right)^2$$

$$= 2k\, S(\underset{\sim}{X})^2$$, where $S(\underset{\sim}{X})$ is centroid size.

Q7. $\underset{\sim}{H} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$

$$\underset{\sim}{H}\,\underset{\sim}{H}^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\underset{\sim}{H}\,\underset{\sim}{H}^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{6}, & -\frac{1}{2}+\frac{1}{6}, & -\frac{1}{2}+\frac{1}{6} \\ -\frac{1}{2}+\frac{1}{6}, & \frac{1}{2}+\frac{1}{6}, & -\frac{1}{2}+\frac{1}{6} \\ -\frac{1}{2}+\frac{1}{6}, & -\frac{1}{2}+\frac{1}{6}, & \frac{1}{2}+\frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -\frac{1}{3} & -\frac{1}{3} \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

as required.

For $k=4$

$$H = \begin{bmatrix} -1/\sqrt{2} & +1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & 3/\sqrt{12} \end{bmatrix}$$

For $k=5$

$$\underset{\sim}{H} = \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & 3/\sqrt{12} & 0 \\ -1/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{20} & -\frac{1}{\sqrt{20}} & \frac{4}{\sqrt{20}} \end{bmatrix}$$

and we can see that $\underset{\sim}{H}\,\underset{\sim}{H}^T = \underset{\sim}{I}_{k-1}$, $\underset{\sim}{H}\,\underset{\sim}{H}^T = \underset{\sim}{C}_k$ using straightforward, but tedious, matrix multiplication.

7. For general $k \geq 3$

Let $\underset{\sim}{h}_j$ be the jth row of $\underset{\sim}{H}$ and so

$$\underset{\sim}{h}_j = \left(-\frac{1}{\sqrt{j(j+1)}}, \ldots, -\frac{1}{\sqrt{j(j+1)}}, \frac{j}{\sqrt{j(j+1)}}, 0, \ldots, 0 \right)$$

Hence,

$$\underset{\sim}{h}_j \underset{\sim}{h}_j^T = \frac{j + j^2}{\sqrt{j(j+1)}\sqrt{j(j+1)}} = 1$$

$$\underset{\sim}{h}_j \underset{\sim}{h}_l^T = \frac{-j + j}{\sqrt{j(j+1)}\sqrt{j(j+1)}} = 0 \quad , j \neq l.$$

Hence, $\underset{\sim}{H}\underset{\sim}{H}^T = \underset{\sim}{I}_{k-1}$.

Let $\underset{\sim}{H}_k$ be the Helmet matrix for k.
We know $\underset{\sim}{H}_3^T \underset{\sim}{H}_3 = \underset{\sim}{C}_3$.
Assume $\underset{\sim}{H}_k^T \underset{\sim}{H}_k = \underset{\sim}{C}_k$ holds.

We will deduce $\underset{\sim}{H}_{k+1}^T \underset{\sim}{H}_{k+1} = \underset{\sim}{C}_{k+1}$,
and hence show the result by induction.

for $\underline{k+1}$.

$$\underset{\sim}{H}_{k+1} = \begin{bmatrix} & \underset{\sim}{H}_k & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \frac{-1}{\sqrt{k(k+1)}} & \cdots & -\frac{1}{\sqrt{k(k+1)}} & \frac{k}{\sqrt{k(k+1)}} \end{bmatrix}$$

So

$\underset{\sim}{H}_{k+1}^T \underset{\sim}{H}_{k+1}$

$$= \begin{bmatrix} & \underset{\sim}{H}_k^T & & \begin{matrix} -1/\sqrt{k(k+1)} \\ \vdots \\ -1/\sqrt{k(k+1)} \end{matrix} \\ 0 & \cdots & 0 & \frac{k}{\sqrt{k(k+1)}} \end{bmatrix} \begin{bmatrix} & \underset{\sim}{H}_k & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \frac{-1}{\sqrt{k(k+1)}} & \cdots & \frac{-1}{\sqrt{k(k+1)}} & \frac{k}{\sqrt{k(k+1)}} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{H}_k^T \underset{\sim}{H}_k + \frac{1}{(\sqrt{k(k+1)})^2} \underset{\sim}{1}\underset{\sim}{1}^T, & \frac{-k}{k(k+1)} \\ \frac{-k}{k(k+1)} & , \frac{k^2}{k(k+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{I}_k - \frac{1}{k}\underset{\sim}{1}\underset{\sim}{1}^T + \frac{1}{k(k+1)}\underset{\sim}{1}\underset{\sim}{1}^T & , & -\frac{1}{k+1} \\ -\frac{1}{k+1} & , & \frac{k}{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{I}_k + \left(\frac{-k-1+1}{k(k+1)}\right)\underset{\sim}{1}\underset{\sim}{1}^T & , & -\frac{1}{k+1} \\ -\frac{1}{k+1} & , & \frac{k}{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{I}_k - \frac{1}{k+1}\underset{\sim}{1}\underset{\sim}{1}^T & , & -\frac{1}{k+1} \\ -\frac{1}{k+1} & , & 1 - \frac{1}{k+1} \end{bmatrix}$$

$$= \underset{\sim}{I}_{k+1} - \frac{1}{k+1}\underset{\sim}{1}_{k+1}\underset{\sim}{1}_{k+1}^T$$

as required.

Hence we have proved the result by induction.