$\boxed{\text{The Uniform Distribution}}$ $U(a,b)$

$f_X(x) = \frac{1}{b-a}$

Relationships

① $X \sim U(0,1) \longrightarrow -\theta \ln(X) \sim \exp(\theta)$

$Y = g(x) = -\theta \ln(x)$ monotone $(0,1)$

$Y^{-1} = g^{-1}(y) = e^{y/-\theta}$

$\frac{d}{dy} Y^{-1} = \frac{-1}{\theta} e^{y/-\theta}$

$f_Y(y) = f_X(e^{-y/\theta}) \left| \frac{-1}{\theta} e^{-y/\theta} \right| = \frac{1}{1-0} \cdot \frac{1}{\theta} e^{-y/\theta} = \frac{1}{\theta} e^{-y/\theta} \equiv \exp(\theta)$

② $X \sim U(\frac{-\pi}{2}, \frac{\pi}{2}) \longrightarrow \tan(X) \sim$ cauchy

Exponential Family

This is not an exponential family as its support depends on the parameters of the model.

MGF

$$M_X(t) = E(e^{tX}) = \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{1}{b-a}\cdot\frac{e^{tx}}{t}\Big|_a^b$$

$$= \frac{e^{tb}}{t(b-a)} - \frac{e^{ta}}{t(b-a)}$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

Sufficient Statistic

$$f_X(\underline{x}|\theta) = \prod_{i}^{n} \frac{1}{b-a} I(a \le x_i \le b)$$

$$= \underbrace{\frac{1}{b-a}}_{h(x)} \underbrace{I(a \le x_{(1)})\, I(x_{(n)} \le b)}_{g(T|a,b)}$$

$\therefore \begin{bmatrix} X_{(1)} \\ X_{(n)} \end{bmatrix}$ is a sufficient statistic

Minimally Sufficient Statistic

$\forall$ 2 sample points $\underline{x}, \underline{y}$

$$\frac{f(\underline{x}|a,b)}{f(\underline{y}|a,b)} = \frac{\frac{1}{b-a} I(a \le x_{(1)}) I(x_{(n)} \le b)}{\frac{1}{b-a} I(a \le y_{(1)}) I(y_{(n)} \le b)} = \frac{I(a \le x_{(1)})\, I(x_{(n)} \le b)}{I(a \le y_{(1)})\, I(y_{(n)} \le b)}$$

This will be a constant function of a and b, iff $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$. Thus $\begin{bmatrix} X_{(1)} \\ X_{(n)} \end{bmatrix}$ is a minimal sufficient statistic.

Ancillary Statistic

$$f_X(x) = \frac{1}{b-a} I(a \le x \le b) \equiv \frac{1}{b-a} f_Z\left(\frac{z}{b-a}\right) \quad w/ \ f_Z(z) = I(0 \le z \le 1)$$

So we have shown the uniform(a,b) to be a scale family of the uniform$(0,1)$. Thus any statistic of the form $\frac{X_i}{X_j}$ is an ancillary statistic.

Complete Statistic

- We can't use the exponential family trick here so we have to find $T(\underline{X}) \ni$ if $E_\theta(g(T)) = 0 \ \forall \theta$ then $P_\theta(g(T)=0) = 1$
- Let's look at $T = X_{(n)}$ from $X_1, X_2, \ldots X_n \sim U(0,b)$

$$E_\theta(g(T)) = \int_0^b g(t) \left[\frac{n!}{(n-n)!(n-1)!} f_X(x)[F_X(x)]^{n-1}[1-F_X(x)]^{n-n}\right] dt$$

$$= \int_0^b g(t)\left[n\left(\frac{1}{b}\right)\left(\frac{t}{b}\right)^{n-1}\right] dt = \int_0^b g(t) \cdot n \cdot b^{-n} \cdot t^{n-1}\, dt$$

$$= nb^{-n} \int_0^b g(t) \cdot t^{n-1}\, dt \qquad = 0 \Rightarrow \frac{d}{db}\, nb^{-n} \int_0^b g(t) \cdot t^{n-1}\, dt$$

$$= \underbrace{\left[nb^{-n}\left(\frac{d}{db}\int_0^b g(t)t^{n-1}\, dt\right)\right]} + \left[\left(\frac{d}{db} nb^{-n}\right)\underbrace{\left(\int_0^b g(t)t^{n-1} dt\right)}_{E(g(T))}\right]$$

$$= nb^{-n} g(b)\, b^{n-1} \quad + \quad 0$$

$$= nb^{-1} g(b) = \frac{n}{b} g(b)$$

$$= 0 \text{ iff } g(b) = 0$$

Method of moments

$m_1 = \frac{1}{n}\sum_i^n x_i$ $\qquad \mu_1 = E(x) = \frac{a+b}{2}$

$m_2 = \frac{1}{n}\sum_i^n x_i^2$ $\qquad \mu_2 = E(x^2) = var(x) + E(x)^2 = \frac{(b-a)^2}{12} + \frac{(b+a)^2}{4}$

$\frac{1}{n}\sum_i^n x_i = \frac{a+b}{2}$ and $\frac{1}{n}\sum_i^n x_i^2 = \frac{(b-a)^2 + 3(b+a)^2}{12} = \frac{b^2 - 2ab + a^2 + 3b^2 + 6ab + 3a^2}{12}$

$\left.\begin{array}{l}\hat{a} = 2\bar{x} - b \\ \hat{b} = 2\bar{x} - a\end{array}\right\}$ Method of moments if the other is known

$= \frac{a^2 + b^2 + ab}{3}$

plug $\hat{a}$ $= \frac{(2\bar{x}-b)^2 + b^2 + b(2\bar{x}-b)}{3}$

$= \frac{4\bar{x}^2 - 4\bar{x}b + b^2 + b^2 + 2\bar{x}b - b^2}{3}$

$= \frac{4\bar{x}^2 + b^2 - 2\bar{x}b}{3}$

Solve numerically.

Maximum Likelihood Estimator,

$L(\theta|\underline{x}) = \prod_i f_x(x) = \left(\frac{1}{b-a}\right)^n I(a \le x_{(1)}) I(x_{(n)} \le b)$

$\ln(L(\theta|\underline{x})) = -n\ln(b-a)$

$\frac{d}{da}\ln(L(\theta|\underline{x})) = \frac{-n}{b-a}\cdot(-1) = \frac{n}{b-a}$, $\frac{d}{db}\ln(L(\theta|\underline{x})) = \frac{-n}{b-a}\cdot 1 = \frac{-n}{b-a}$

$\frac{d^2}{da^2}\ln(L(\theta|\underline{x})) = \frac{-n}{(b-a)^2}\cdot -1 = \frac{n}{b-a} > 0$, $\frac{d}{db}\ln(L(\theta|\underline{x})) = \frac{n}{(b-a)^2}\cdot 1 = \frac{n}{(b-a)^2} \ge 0$

$\hat{a} = x_{(1)}$ $\qquad \hat{b} = x_{(n)}$