$\boxed{\text{The Gamma Distribution}}$ gamma(α, β)

$$f_X(x) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\, x^{\alpha-1}\, e^{-x/\beta}\, I(x, \alpha, \beta > 0)$$

① $X \sim \text{gamma}(\alpha, \beta) \longrightarrow 2X/\beta \sim \chi^2_{(2\alpha)}$

$Y = g(x) = \frac{2}{\beta} X$

$Y^{-1} = g^{-1}(Y) = \frac{\beta}{2} Y$

$\frac{d}{dY} g^{-1}(Y) = \frac{\beta}{2}$

$$f_Y(y) = f_X\left(\tfrac{\beta}{2} Y\right)\left|\tfrac{\beta}{2}\right| = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(\tfrac{\beta}{2} y\right)^{\alpha-1} e^{-\frac{\beta}{2}Y/\beta} \left|\tfrac{\beta}{2}\right|$$

$$= \frac{1}{\Gamma(\alpha)\,2^{\alpha}}\, Y^{\alpha-1}\, e^{-Y/2} \longrightarrow \text{gamma}(\alpha, 2) \equiv \chi^2(2\alpha)$$

② $X_1, X_2, \ldots X_n \overset{ind}{\sim} \text{gamma}(\alpha_i, \beta) \longrightarrow \Sigma X_i \sim \text{gamma}(\Sigma \alpha_i, \beta)$

③ $\text{gamma}(1, \beta) \equiv \text{exponential}(\beta)$

④ $X_1, X_2, \ldots X_n \overset{iid}{\sim} \text{gamma}\left(\frac{v_i}{2}, 2\right) \equiv \chi^2_{(v_i)} \longrightarrow \sum_1^n X_i \sim (\Sigma v_i)$

MGF

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}\left[\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\right]dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-x/\left(\beta/(1-\beta t)\right)} dx$$

$$= \frac{\Gamma(\alpha)\left(\beta/(1-\beta t)\right)^\alpha}{\Gamma(\alpha)\beta^\alpha}\underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha)\left(\beta/(1-\beta t)\right)^\alpha} x^{\alpha-1} e^{-x/\left(\beta/(1-\beta t)\right)} dx}_{\text{gamma}\left(\alpha,\left(\beta/(1-\beta t)\right)\right)}$$

$$= \frac{1}{\beta^\alpha}\left(\frac{\beta}{1-\beta t}\right)^\alpha$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha$$

Sufficient Statistic

$$f(\underline{x}|\alpha,\beta) = f(\underline{x}|\theta) = \prod_1^n f(x|\theta) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \prod_1^n x_i^{\alpha-1} e^{\frac{-1}{\beta}\sum_1^n x_i}$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n e^{\frac{-1}{\beta}\sum_1^n x_i} \prod_1^n x_i^{\alpha-1}$$

$$= \underbrace{\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n e^{\frac{-1}{\beta}\sum_1^n x_i + (\alpha-1)\sum_1^n \ln(x_i)}} \underbrace{(1)}_{h(x)}$$

$$\therefore T(\underline{x}) = \begin{bmatrix} \sum_1^n x_i \\ \sum_1^n \ln(x_i) \end{bmatrix} \text{ is a sufficient statistic}$$

Minimally Sufficient

$\forall$ 2 sample points $\underline{x}, \underline{y}$

$$\frac{f(\underline{x}|\theta)}{f(\underline{y}|\theta)} = \frac{\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n e^{\frac{-1}{\beta}\sum_1^n x_i + (\alpha-1)\sum_1^n \ln(x_i)}}{\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n e^{\frac{-1}{\beta}\sum_1^n y_i + (\alpha-1)\sum_1^n \ln(y_i)}} = e^{\frac{-1}{\beta}(\Sigma x_i - \Sigma y_i) + (\alpha-1)(\Sigma \ln(x) - \ln(y))}$$

* This will be a constant of α, β iff $\Sigma x_i = \Sigma y_i$ and $\Sigma \ln(x) = \Sigma \ln(y)$

$\therefore \begin{bmatrix} \sum_1^n x_i \\ \sum_1^n \ln(x) \end{bmatrix}$ is a minimally sufficient statistic for α, β

Ancillary Statistic

$X_1, X_2, \ldots X_n \sim \text{gamma}(\alpha_0, \beta)$

$$f_x(x) = \frac{1}{T(\alpha_0)\beta^{\alpha_0}} x^{\alpha_0 - 1} e^{-x/\beta} \equiv \frac{1}{\beta} f_z\left(\frac{z}{\beta}\right) \text{ where } f_z(z) = \frac{1}{T(\alpha_0)} z^{\alpha_0 - 1} e^{-z}$$

Thus we have shown gamma to be a scale-family so any statistic that is a function of a ratio of the form $\frac{X_i}{X_j}$ will be ancillary.

Complete Statistic

$$f_x(x) = \frac{1}{T(\alpha_0)\beta^{\alpha_0}} x^{\alpha_0 - 1} e^{-x/\beta} = \underbrace{\frac{1}{T(\alpha_0)\beta^{\alpha_0}}}_{c(\alpha,\beta)} e^{(\alpha_0 - 1)\ln(x) - x/\beta} \underbrace{(1)}_{h(x)}$$

$w_1(a,b) = \alpha_0 - 1 \quad w_2 = \frac{-1}{\beta}$

$t_1(x) = \ln(x) \quad t_2(x) = x$

$\therefore$ The gamma function is an exponential family so a complete statistic for α, β is $\begin{bmatrix} \Sigma \ln(x_i) \\ \Sigma x \end{bmatrix}$?

Method of Moments

$m_1 = \frac{1}{n}\sum x_i$ $\quad \mu_1 = E(x) = \alpha\beta$

$m_2 = \frac{1}{n}\sum x_i^2$ $\quad \mu_2 = E(x^2) = var(x) + E(x)^2 = \alpha\beta^2 + \alpha^2\beta^2$

Solve $\bar{x} = \alpha\beta$

$\frac{1}{n}\sum x^2 = \alpha\beta^2(1-\alpha)$ $\Rightarrow \beta = \alpha\bar{x} \Rightarrow \frac{1}{n}\sum x^2 = \alpha^2\bar{x}^2(1-\alpha)$

$\Rightarrow n\bar{x}^2 = \alpha^2\bar{x}^2(1-\alpha) \Rightarrow n = \alpha^2 - \alpha^3$

$\vdots$ This gets too nasty (non linear)

- Let's say α is known, let's find the Method of Moments estimator for β

$m_1 = \frac{1}{n}\sum_1^n x_i$ $\quad \mu_1 = E(x) = \alpha_0\beta$

$\hat{\beta} = \bar{x}/\alpha_0$

- Let's say β is known, let's find the Method of Moments estimator for α

$m_1 = \frac{1}{n}\sum_1^n x_i$ $\quad \mu_1 = E(x) = \alpha\beta_0$

$\hat{\alpha} = \bar{x}/\beta_0$

Maximum Likelihood Estimator

$L(\alpha,\beta|\underline{x}) = \prod_1^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \prod_1^n x^{\alpha-1} e^{-\frac{1}{\beta}\sum_1^n x_i}$

$\ln(L(\alpha,\beta|\underline{x})) = -n\ln(\Gamma(\alpha)\beta^\alpha) + (\alpha-1)\sum_1^n \ln(x_i) - \frac{1}{\beta}\sum_1^n x_i = -n\ln(\Gamma(\alpha)) - \alpha n\ln(\beta) + (\alpha-1)\sum_1^n \ln(x_i) - \frac{1}{\beta}\sum_1^n x_i$

$\frac{\partial}{\partial\alpha}\ln(L(\alpha,\beta|\underline{x})) = \frac{-n\Gamma'(\alpha)}{\Gamma(\alpha)} - n\ln(\beta) + \sum_1^n \ln(x_i) \overset{set}{=} 0$ $\leftarrow$ Plug this in and solve numerically

$\frac{\partial}{\partial\beta}\ln(L(\alpha,\beta|\underline{x})) = \frac{-\alpha n}{\beta} + \frac{\sum_1^n x_i}{\beta^2} \overset{set}{=} 0 \Rightarrow \frac{\alpha n}{\beta} = \frac{\sum_1^n x_i}{\beta^2} \Rightarrow \hat{\beta} = \bar{x}/\alpha$

$\frac{\partial^2}{\partial\alpha^2}\ln(L(\alpha,\beta|\underline{x})) =$ really ugly

$$\frac{\partial^2}{\partial\beta^2}\ln(L(\alpha,\beta|\underline{x})) = \frac{\alpha n}{\beta^2} - \frac{2\sum_1^n x_i}{\beta^3}\bigg|_{\bar{x}/\alpha} = \frac{\alpha n}{(\bar{x}/\alpha)^2} - \frac{2\sum_1^n x_i}{(\bar{x}/\alpha)^3} \Rightarrow \frac{\alpha^3 n^3}{(\sum_1^n x_i)^2} - \frac{2\alpha^3 n^3}{(\sum_1^n x_i)^2} < 0$$

$\hat{\beta} = \bar{x}/\alpha$

α needs to be solved numerically