Chapter 9 : Interval Estimation

Definition 9.1.1 An interval estimate of a real valued parameter θ is any pair of functions $L(\underline{x})$ and $U(\underline{x})$ that satisfy $L(\underline{x}) \leq U(\underline{x})\ \forall\ \underline{x} \in \mathcal{X}$
This gives interval estimator $[L(\underline{X}), U(\underline{X})]$

Example 9.1.2 $X_1, X_2, X_3, X_4 \sim N(\mu, 1)$
A simple interval estimate of μ is $[\bar{X}-1, \bar{X}+1]$

Example 9.1.3 Note $P(\mu = \bar{X}) = 0$
But $P(\bar{X}-1 \leq \mu \leq \bar{X}+1)$
$= P(-2 \leq \frac{\bar{X}-\mu}{\sqrt{1/4}} \leq 2)$
$= P(-2 \leq Z \leq 2)$
$= .9544$

Definition 9.1.4 Coverage Probability = Probability that the confidence interval contains the population parameter
$= P(\theta \in [L(\underline{X}), U(\underline{X})] \mid \theta)$

Definition 9.1.5 Confidence Coefficient the infimum of coverage probabilities
$= \inf_{\theta} P(\theta \in [L(\underline{x}), U(\underline{x})] \mid \theta)$

* Here the interval is the RV

Example 9.1.6 $X_1, X_2, \ldots X_n \sim \text{Uniform}(0,\theta)$

$Y = X_{(n)}$

consider ① $[aY, bY]$ $\quad 1 \le a < b$

② $[Y+c, Y+d]$ $\quad 0 \le c < d$

For ① $P_\theta(\theta \in [aY, bY]) = P_\theta(aY \le \theta \le bY) = (a \le \frac{\theta}{Y} \le b)$

$= P_\theta(\frac{1}{b} \le \frac{Y}{\theta} \le \frac{1}{a})$

$= P_\theta(\frac{1}{b} \le T \le \frac{1}{a}) \qquad T = Y/\theta$

$= \int_{1/b}^{1/a} nt^{n-1}\,dt = (\frac{1}{a})^n - (\frac{1}{b})^n$

$\uparrow$ as $f_T(t) = nt^{n-1}$

For ② $P_\theta(\theta \in [Y+c, Y+d]) = P_\theta(Y+c \le \theta \le Y+d)$

$= P_\theta(\frac{Y}{\theta} + \frac{c}{\theta} \le 1 \le \frac{Y}{\theta} + \frac{d}{\theta})$

$= P_\theta(\frac{c}{\theta} \le 1 - \frac{Y}{\theta} \le \frac{d}{\theta})$

$= P_\theta(\frac{c}{\theta} - 1 \le -\frac{Y}{\theta} \le \frac{d}{\theta} - 1)$

$= P_\theta(1 - \frac{d}{\theta} \le \frac{Y}{\theta} \le 1 - \frac{c}{\theta})$

$= P_\theta(1 - \frac{d}{\theta} \le T \le 1 - \frac{c}{\theta})$

$= \int_{1-d/\theta}^{1-c/\theta} nt^{n-1}\,dt = (1 - \frac{c}{\theta})^n - (1 - \frac{d}{\theta})^n$

Note $\lim_{\theta \to \infty} (1-\frac{c}{\theta})^n - (1-\frac{d}{\theta})^n = 0$

9.2 Finding Estimators

Inverting a Test Statistic Remember hypothesis testing?

Example 9.2.1 $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Consider: $H_0: \mu = \mu_0$

$H_a: \mu \ne \mu_0$

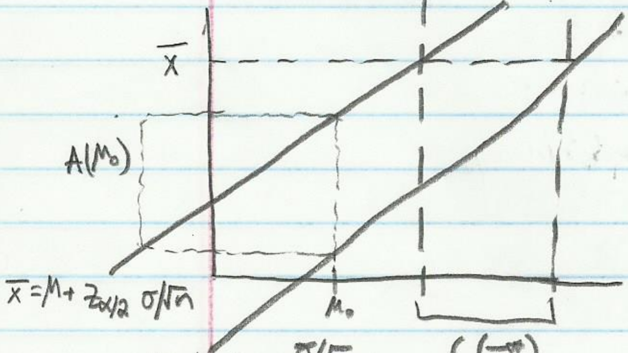
$R = \{\underline{x} \mid |\bar{x} - \mu_0| > (z_{\alpha/2}\sigma)/\sqrt{n}\}$ = Rejection Region

$A = \{\underline{x} \mid |\bar{x} - \mu_0| \le (z_{\alpha/2}\sigma)/\sqrt{n}\}$ = Acceptance Region

$\equiv \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \forall \mu_0$

$P(H_0 \text{ is accepted} \mid \mu = \mu_0) = 1 - \alpha$

$\therefore P_\mu(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\sigma/\sqrt{n}) = 1 - \alpha$



Theorem 9.2.2 $\forall\ \theta_0 \in \Theta$ let $A(\theta_0)$ be the acceptance region of a level α test of $H_0: \theta = \theta_0$. $\forall\ \underline{x} \in \mathcal{X}$ define $C(\underline{x}) = \{\theta_0 \mid \underline{x} \in A(\theta_0)\}$

1) If $A(\theta_0)$ is a level α test $C(\underline{x})$ is a $1-\alpha$ CI
2) If $C(\underline{x})$ is a $1-\alpha$ CI $A(\theta_0)$ is the acceptance region of a level α test

Proof pg. 422

Note: These are families of tests one $\forall\ \theta_0$

Example 9.23 $\exp(\lambda)$ population

$H_0: \lambda = \lambda_0$

$H_a: \lambda \neq \lambda_0$

$$LRT = \left(\frac{\Sigma x_i}{n\lambda_0}\right)^n e^n e^{-\Sigma x_i/\lambda_0}$$

Thus, $A = \left\{ x \,\middle|\, \left(\frac{\Sigma x_i}{\lambda_0}\right)^n e^{-\Sigma x_i/\lambda_0} \geq k^* \right\}$

Choose $k \ni P_{\lambda_0}(\underline{X} \in A) = 1-\alpha$

Thus, $C(\underline{x}) = \left\{ \lambda \,\middle|\, \left(\frac{\Sigma x_i}{\lambda}\right)^n e^{-\Sigma x_i/\lambda} \geq k^* \right\}$

$\vdots$ Numerical Solution

$P_\lambda\left(\frac{1}{5.48}\Sigma x_i \leq \lambda \leq \frac{1}{.441}\Sigma x_i\right) = .90$

Example 9.2.4 $X_1, X_2, \ldots X_n \sim N(\mu, \sigma^2)$

CI: $(-\infty, U(\underline{x})] \leftrightarrow$ Use $H_0: \mu = \mu_0$ vs $H_a: \mu < \mu_0$ test

LRT rejects H_0 if $\frac{\bar{x} - \mu_0}{s/\sqrt{n}} < -t_{(n-1,\alpha)}$

$A = \bar{x} + t_{(n-1,\alpha)}(s/\sqrt{n}) \geq \mu_0$

Thus $C(\underline{x}) = \{\underline{x} \in A\} = \{\mu_0 \mid \bar{x} + t_{(n-1,\alpha)}(s/\sqrt{n}) \geq \mu_0\}$

$= (-\infty, \bar{x} + t_{(n-1,\alpha)}(s/\sqrt{n})]$ is a $1-\alpha$ confidence int.

Example 9.2.5 $X_1, \dots, X_n \sim \text{Bernoulli}(p)$

- Look for CI $(L(\underline{x}), 1]$ $1-\alpha$ CI
- Consider $H_0: p = p_0$

  $H_a: p > p_0$

  $T = \sum X_i \sim \text{binomial}(n, p)$
- Reject H_0 if $T > k(p_0)$ choose $k(p_0) \ni$ we have a level α test

$$\sum_{y=0}^{k(p_0)} \binom{n}{y} p_0^y (1-p_0)^{n-y} \geq 1-\alpha$$

$$\&$$

$$\sum_{y=0}^{k(p_0)-1} \binom{n}{y} p_0^y (1-p_0)^{n-y} < 1-\alpha$$

- $A = \{ t \mid t \leq k(p_0) \}$
- $C(t) = \{ p_0 \mid t \leq k(p_0) \}$

  $= \{ p_0 \mid p_0 \geq k^{-1}(t) \}$

  $= (k^{-1}(T), 1]$

  $\ni, \ k^{-1}(T) = \sup \left(p \mid \sum_{0}^{t-1} \binom{n}{y} p^y (1-p)^{n-y} \geq 1-\alpha \right)$

Pivotal Quantities – A RV whose distribution doesn't depend on the parameter

Definition 9.2.6 $Q(\underline{X}, \theta)$ is a pivotal Quantity if the distribution of $Q(\underline{X}, \theta)$ is ind. of all parameters

That is if $\underline{X} \sim F(\underline{x} \mid \theta)$ then $Q(\underline{X}, \theta)$ has the same distribution $\forall\ \theta$.

Example 9.2.7

Form of pdf	Type of pdf	Pivotal Quant.
$f(x-\mu)$	Location	$\bar{X} - \mu$
$\frac{1}{\sigma} f(\frac{x}{\sigma})$	Scale	$\bar{X}/\sigma$
$\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$	Location-Scale	$\frac{\bar{X}-\mu}{S}$

$X_1, X_2, \dots X_n \sim N(\mu, \sigma^2) \rightarrow t = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ is a pivot

Example 9.2.8 $X_1, X_2, \ldots X_n \overset{iid}{\sim} \exp(\lambda)$

$T = \sum X_i$ is a sufficient for λ

$T \sim \text{gamma}(n, \lambda)$

$Q(T,\lambda) = \frac{2T}{\lambda} \sim \text{gamma}(n, 2)$

$\therefore$ T is a pivot

In general, If $f(t|\theta) = g(Q(t,\theta)) \left| \frac{\partial}{\partial t} Q(t,\theta) \right|$ then $Q(T,\theta)$ is a pivot

$P_\theta(a \le Q(\underline{X}, \theta) \le b) \ge 1-\alpha$ then $\forall\ \theta_0 \in \Theta$

$A(\theta_0) = \{x \mid a \le Q(\underline{x}, \theta_0) \le b\}$ is the level α acceptance region for $H_0: \theta = \theta_0$

$C(\underline{x}) = \{\theta_0 \mid a \le Q(\underline{x}, \theta_0) \le b\}$

$= \{L(x,b), U(x,a)\}$ dec.

$= \{L(x,a), U(x,b)\}$ inc

Example 9.2.9 Continuing 9.2.8

$T = \sum X_i$

$Q(T,\lambda) = \frac{2T}{\lambda} \sim \chi^2_{2n}$

Choose a, b to satisfy $P(a \le \chi^2_{2n} \le b) = 1-\alpha$

Then $P_\lambda(a \le \frac{2T}{\lambda} \le b) = P_\lambda(a \le Q(T,\lambda) \le b) = P(a \le \chi^2_{2n} \le b) = 1-\alpha$

$A = \{t \mid a \le \frac{2t}{\lambda} \le b\} \rightarrow C(t) = \{\lambda \mid \frac{2t}{b} \le \lambda \le \frac{2t}{a}\} \leftarrow 1-\alpha$ conf. int.

Example 9.2.10 $X_1, X_2, \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is a pivot

CI: $\{\mu \mid \bar{x} - a\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + a\,\sigma/\sqrt{n}\}$ $-\sigma^2$ known

$\{\mu \mid \bar{x} - t_{(n-1,\alpha/2)}\, s/\sqrt{n} \le \mu \le \bar{x} + t_{(n-1,\alpha/2)}\, s/\sqrt{n}\}$ $-\sigma^2$ unknown

$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ is also a pivot

CI $\left\{ \sigma^2 \,\middle|\, \frac{(n-1)s^2}{b} \le \sigma^2 \le \frac{(n-1)s^2}{a} \right\}$

$a = \chi^2_{(n-1, 1-\alpha/2)}$

$b = \chi^2_{(n-1, \alpha/2)}$

Pivoting the CDF

Theorem 9.2.12 · Let T be a statistic w/ continuous cdf $F_T(t|\theta)$

· Let $\alpha_1 + \alpha_2 = \alpha$

i) If $F_T(t|\theta)$ is a <u>decreasing</u> function of θ $\forall t$

$\theta_L(t)$ and $\theta_u(t)$ can be defined by

$F_T(t|\theta_u(t)) = \alpha_1$ & $F_T(t|\theta_L(t)) = 1-\alpha_2$

ii) If <u>increasing</u>

$F_T(t|\theta_u(t)) = 1-\alpha_2$ & $F_T(t|\theta_u(t)) = \alpha_1$

Then $[\theta_L(T), \theta_u(t)]$ is a $1-\alpha$ CI for θ

<u>Proof</u> p432

Example 9.2.13 $X_1, X_2, \ldots X_n \overset{iid}{\sim} e^{-(x-\mu)}$

$Y = X_{(1)}$ is sufficient for μ

$f_Y(y|\mu) = ne^{-n(y-\mu)}$

Fix α and define U & L

$\int_{\mu U(y)}^{y} ne^{-n(u-\mu U(y))}\,du = \alpha/2$, $\int_{y}^{\infty} ne^{-n(u-\mu_L(y))}\,du = \alpha/2$

thus $1-e^{-n(y-\mu U(y))} = \alpha/2$, $e^{-n(y-\mu L(y))}\,du = \alpha/2$

$\mu U(y) = y + \frac{1}{n}\log(1-\alpha/2)$, $\mu_L(y) = y + \frac{1}{n}\log(\alpha/2)$

$\therefore$ $C(Y) = \left\{\mu \,\middle|\, Y + \frac{1}{n}\log\left(\frac{\alpha}{2}\right) \le \mu \le Y + \frac{1}{n}\log(1-\alpha/2)\right\}$ is a $1-\alpha$ CI

Theorem 9.2.14 · Let T be a discrete statistic w/ CDF $F_T(t|\theta) = P(T \le t|\theta)$

· Let $\alpha_1 + \alpha_2 = \alpha$

· Suppose $\forall t \in T$, $\theta_L(t), \theta_u(t)$ can be defined as follows

i) $F_T(t|\theta)$ decreasing

$P(T \le t|\theta_u(t)) = \alpha_1$, $P(T \ge t|\theta_L(t)) = \alpha_2$

ii) $F_T(t|\theta)$ increasing

$P(T \ge t|\theta_u(t)) = \alpha_1$, $P(T \le t|\theta_L(t)) = \alpha_2$

Then $[\theta_L(T), \theta_u(T)]$ is a $1-\alpha$ Confid. Int

<u>Proof</u>: p434

Example 9.2.15 $X_1, X_2, \ldots X_n \sim \text{Poisson}(\lambda)$

$Y = \Sigma X_i$ is sufficient for λ

$Y \sim \text{Poisson}(n\lambda)$

$\alpha_1 = \alpha_2 = \alpha/2$

- If $Y = y_0$ is observed
- Solve

$$\sum_{0}^{y_0} e^{-n\lambda} \frac{(n\lambda)^k}{k!} = \frac{\alpha}{2} \quad \text{and} \quad \sum_{y_0}^{\infty} e^{-n\lambda} \frac{(n\lambda)^k}{k!} = \frac{\alpha}{2}$$

⋮

- CI: $\left\{ \lambda \,\middle|\, \frac{1}{2n} \chi^2_{2y_0, 1-\alpha/2} \le \lambda \le \frac{1}{2n} \chi^2_{2(y_0+1), \alpha/2} \right)$

Bayes Intervals For posterior distribution $\pi(\theta|\underline{x})$

Credible Probability for A $\int_A \pi(\theta|\underline{x}) d\theta$

Example 9.2.16 $X_1, X_2, \ldots X_n \sim \text{Poisson}(\lambda)$

$\lambda \sim \text{gamma}(a, b)$

$\pi(\lambda | \Sigma X = \Sigma x_i) = \text{gamma}(a + \Sigma x, [n + (1/b)]^{-1})$

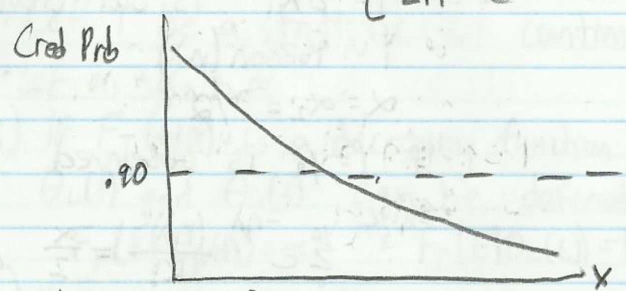
$$\frac{2(nb+1)}{b} \lambda \sim \chi^2_{2(a + \Sigma x_i)}$$

Thus a $1-\alpha$ CI is

$$\left\{ \lambda \,\middle|\, \frac{b}{2(nb+1)} \chi^2_{2(\Sigma x_i + a), 1-\alpha/2} \le \lambda \le \frac{b}{2(nb+1)} \chi^2_{2(\Sigma x + a), \alpha/2} \right\}$$

Credible Prb	vs Coverage Prob.
Assuming prior knowledge	uncertainty in sampling
from posterior	from repeated experiments
$1-\alpha$% sure of coverage	$1-\alpha$% coverage in unbounded trials

Example 9.2.17 from 9.2.15 $\left\{ \underbrace{\frac{1}{2n}\chi^2_{(2\Sigma x, 1-\alpha/2)}}^{L}, \overbrace{\frac{1}{2n}\chi^2_{(2(\Sigma x+1), \alpha/2)}}^{U} \right\}$



As $\sum_i x_i \to \infty$ $P(L \le \lambda \le U) \to 0$

Thus this CI cannot maintain a nonzero credible probability

Example 9.2.18
- $X_1, X_2, \ldots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$
- $\theta \sim N(\mu, \tau^2)$
- $\pi(\theta|\bar{x}) \sim N(\delta^B(\bar{x}), var(\theta|\bar{x}))$

  w/ $\delta^B(\bar{x}) = \frac{\sigma^2}{\sigma^2+n\tau^2}\mu + \frac{n\tau^2}{\sigma^2+n\tau^2}\bar{x}$

  and $var(\theta|\bar{x}) = \frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}$

- $\frac{\theta - \delta^B(\bar{x})}{\sqrt{var(\theta|\bar{x})}}$ (Posterior)

- $1-\alpha$ CI: $\left(\delta^B(\bar{x}) \pm Z_{\alpha/2}\sqrt{var(\theta|\bar{x})}\right)$ (for θ)

  CREDIBLE SET ↑

Example 9.3.1 $X_1, X_2, \ldots X_n \sim N(\mu, \sigma^2)$

$Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is a pivot $N(0,1)$

$P(a \le Z \le b)\ 1-\alpha$

$1-\alpha$ Conf. Int $\left(\bar{x} - b\frac{\sigma}{\sqrt{n}}, \bar{x} - a\frac{\sigma}{\sqrt{n}}\right)$

- What's the best choice for a and b?

  We need to satisfy $P(a \le Z \le b) = 1-\alpha$ and minimize $b-a$

Theorem 9.3.2 Let $f(x)$ be a unimodal pdf. If $[a,b]$ satisfies
i) $\int_a^b f(x)\,dx = 1-\alpha$
ii) $f(a) = f(b) > 0$ and
iii) $a \le x^* \le b$ where x^* is a mode of $f(x)$
Then $[a,b]$ is the shortest among all intervals satisfying (i)
Proof 442

Example 9.3.3 For normal based intervals

$$\bar{x} - b\frac{s}{\sqrt{n}} \le \mu \le \bar{x} - a\frac{s}{\sqrt{n}}$$

$$a = -t_{(n-1,\alpha/2)} \quad \& \quad b = t_{(n-1,\alpha/2)}$$

$$\left.\begin{array}{l} Length(s) = (b-a)\, s/\sqrt{n} \\ E(Length(s)) = (b-a)\, c(n)\, \sigma/\sqrt{n} \end{array}\right\} \xrightarrow{9.3.2} \begin{array}{l} a = -t_{(n-1,\alpha/2)} \\ b = t_{(n-1,\alpha/2)} \end{array} \text{ are optimal}$$

Example 9.3.4
- $X \sim gamma(k, \beta)$
- $Y = X/\beta$ is a pivot
- $Y \sim gamma(k, 1)$
- $P(a \le Y \le b) = 1-a$
- $\{\beta \mid \frac{x}{b} \le \beta \le \frac{x}{a}\}$
- length $= (\frac{1}{a} - \frac{1}{b})x \propto (\frac{1}{a} - \frac{1}{b})$
- minimize $\frac{1}{a} - \frac{1}{b(a)}$ w/ respect to a subject to $\int_a^{b(a)} f_Y(y)\,dy = 1-\alpha$