Chapter 8

Definition 8.1.1 A hypothesis is a statement about a population parameter

Definition 8.1.2 The two complementary hypotheses in a hypothesis testing problem are called the null hypothesis and alternative hypothesis. (H_0 and H_1)

Population parameter θ

null hypothesis: $H_0: \theta \in \Theta_0$ $\ni$ Θ_0 is called the null space

alternative hypothesis: $H_1: \theta \in \Theta_0^c$ $\ni$ Θ_0^c is the complement of the null space

① $H_0: \theta = \theta_0$, $H_1: \theta \neq \theta_0$ ② $H_0: \theta \leq \theta_0$, $H_1: \theta > \theta_0$ ③ $H_0: \theta \geq \theta_0$, $H_1: \theta < \theta_0$ ④ $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$

Definition 8.1.3 A hypothesis testing procedure or hypothesis test is a rule that specifies

1) For which sample values the decision is made to accept H_0 as true (fail to reject)
2) For which sample values H_0 is rejected and H_1 is accepted as true (reject)

Typically a hypothesis test is specified in terms of a test statistic $W(X_1, X_2, \ldots X_n) = W(\underline{X})$. ie $\bar{X} = \frac{1}{n}\sum_1^n X < 3$ reject → here, $W(\underline{X}) = \frac{1}{n}\sum_1^n X = \bar{X}$ is the test statistic

Definition 8.2.1 Likelihood Ratio Test for testing $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_0^c$ is

$$\lambda(\underline{x}) = \frac{\sup_{\Theta_0} L(\theta|x)}{\sup_{\Theta} L(\theta|x)} = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} \quad \ni \hat{\theta}_0 = \text{MLE over } \Theta_0, \; \hat{\theta} = \text{MLE over } \Theta$$

Rejection Region: $R = \{\underline{x} \mid \lambda(\underline{x}) \leq c\}$ where $c \in [0,1]$

Example 8.2.2 $X_1, X_2, \dots X_n \sim N(\theta, 1)$

$$H_0: \theta = \theta_0$$
$$H_1: \theta \neq \theta_0$$

$$\lambda(\underline{x}) = \frac{L(\theta_0 \mid x)}{L(\bar{x} \mid x)} = \frac{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_1^n (x_i - \theta_0)^2}}{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_1^n (x_i - \bar{x})^2}}$$

$$= e^{\frac{1}{2}\left(\sum_1^n (x_i - \bar{x})^2 - \sum_1^n (x_i - \bar{x})^2\right)}$$

$$= e^{-\frac{1}{2} n (\bar{x} - \theta_0)^2}$$

$$\therefore R = \{x \mid e^{-\frac{1}{2} n (\bar{x} - \theta_0)^2} \leq c\}$$
$$= \{x \mid |\bar{x} - \theta_0| \geq \sqrt{-2(\log(c))/n}\}$$

* Thus we reject H_0 when the distance between $\bar{x}$ and θ_0 is too large

Example 8.2.3 Let $X_1, X_2, \dots X_n$ be a random sample from an exponential population $f(x \mid \theta) = e^{-(x-\theta)} I(x \geq \theta)\, I(-\infty < \theta < \infty)$

$$L(\theta \mid \underline{x}) = e^{-\sum (x_i - \theta)} I(x_{(1)} \geq \theta)$$

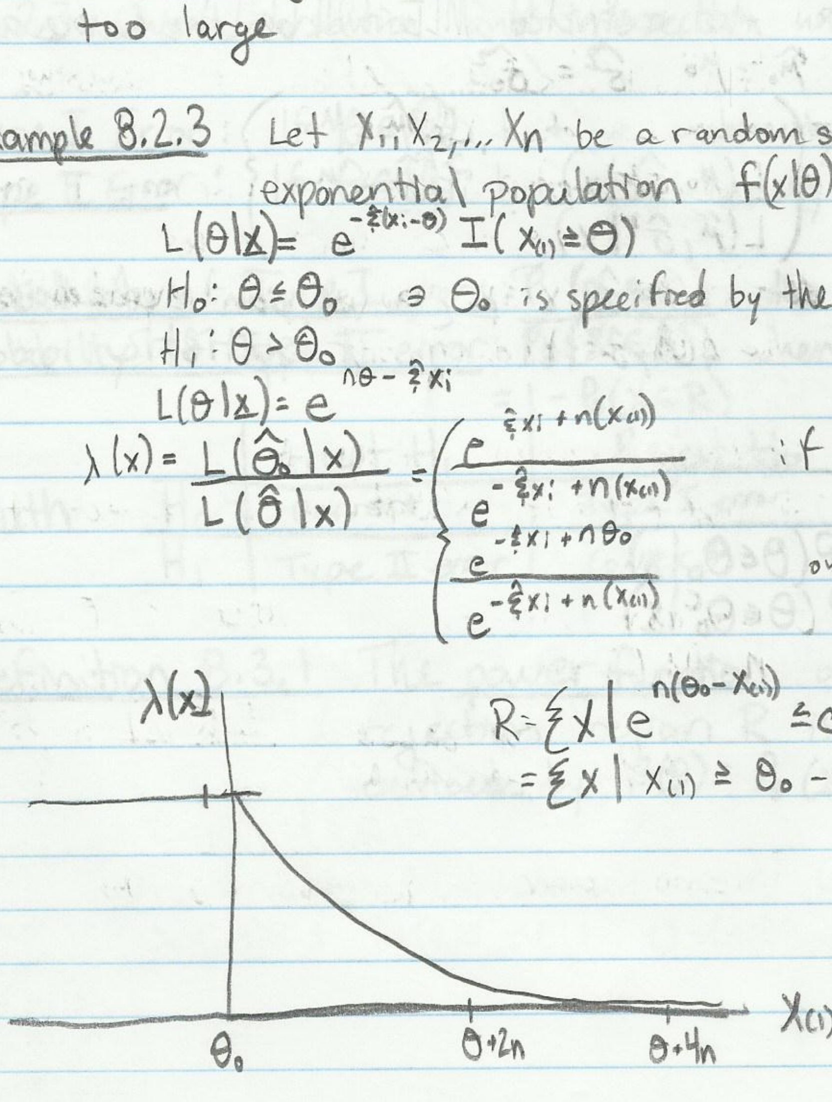
$H_0: \theta \leq \theta_0 \Rightarrow \theta_0$ is specified by the experimenter

$H_1: \theta > \theta_0$

$$L(\theta \mid \underline{x}) = e^{n\theta - \sum x_i}$$

$$\lambda(x) = \frac{L(\hat{\theta}_0 \mid x)}{L(\hat{\theta} \mid x)} = \begin{cases} \dfrac{e^{\sum x_i + n(x_{(1)})}}{e^{-\sum x_i + n(x_{(1)})}} & \text{if } x_{(1)} \leq \theta_0 \\[2ex] \dfrac{e^{-\sum x_i + n\theta_0}}{e^{-\sum x_i + n(x_{(1)})}} & \text{ow} \end{cases}$$

$$R = \{x \mid e^{n(\theta_0 - x_{(1)})} \leq c\}$$
$$= \{x \mid x_{(1)} \geq \theta_0 - \tfrac{\log(c)}{n}\}$$



Theorem 8.2.4 If $T(\underline{X})$ is a sufficient statistic for θ and $\lambda^*(t)$ and $\lambda(x)$ are the likelihood ratio test statistics based on T and X then $\lambda(T(\underline{x})) = \lambda(\underline{x})$ $\forall\ \underline{x}$ in the sample space

proof 377

Example 8.2.6 Suppose $X_1, X_2, \ldots X_n$ are a random sample from $N(\mu, \sigma^2)$

$H_0: \mu \leq \mu_0$

$H_1: \mu > \mu_0$

$$\lambda(\underline{x}) = \frac{\sup_{(\mu,\sigma^2 \mid \mu \leq \mu_0, \sigma^2 > 0)} L(\mu, \sigma^2 \mid x)}{\sup_{(\mu,\sigma^2 \mid \mu > \mu_0, \sigma^2 > 0)} L(\mu, \sigma^2 \mid x)} = \frac{\sup_{(\mu,\sigma^2 \mid \mu \leq \mu_0, \sigma^2 > 0)} L(\mu, \sigma^2 \mid x)}{L(\hat{\mu}, \hat{\sigma}^2 \mid x)}$$

- If $\hat{\mu} \leq \mu_0$ the restricted MLE will be the same as the unrestricted

otherwise $\hat{\mu}_0 = \mu_0$ $\hat{\sigma}^2 = \hat{\sigma}_0^2$

$$\therefore \lambda(\underline{x}) = \begin{cases} 1 & \text{if } \hat{\mu} \leq \mu_0 \\ \dfrac{L(\mu_0, \hat{\sigma}_0^2 \mid x)}{L(\hat{\mu}, \hat{\sigma}^2 \mid x)} & \text{if } \hat{\mu} > \mu_0 \end{cases}$$

Note: set $\dfrac{L(\mu_0, \hat{\sigma}_0^2 \mid x)}{L(\hat{\mu}, \hat{\sigma}^2 \mid x)} \leq c$ we can rewrite this test as a t statistic.

Bayesian Tests

$P(H_0 \text{ is true} \mid \underline{x}) = P(\theta \in \Theta_0 \mid \underline{x})$

$P(H_1 \text{ is true} \mid \underline{x}) = P(\theta \in \Theta_0^c \mid \underline{x})$

Example 8.2.7 $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, $\pi(\theta) \sim N(\mu, \tau^2)$
where σ^2, μ, τ^2 are known
$H_0: \theta \le \theta_0$
$H_1: \theta > \theta_0$

$$\pi(\theta|\underline{X}) \sim N\left(\frac{n\tau^2\bar{x} + \sigma^2\mu}{n\tau^2 + \sigma^2}, \frac{\sigma^2\tau^2}{n\tau^2 + \sigma^2}\right)$$

$$P(\theta \in \Theta_0 | X) = P(\theta \le \theta_0 | \underline{X})$$
$$P(\theta \in \Theta_0^c | X) = 1 - P(\theta \le \theta_0 | \underline{X})$$

An intuitive rule would be to fail to reject iff $\frac{1}{2} \le P(\theta \in \Theta_0 | \underline{X})$ or $P(\theta \in \Theta_0 | \underline{X}) = P(\theta \le \theta_0 | \underline{X})$

8.2.3 Union-Intersection and Intersection union tests Omitted

Type I Error: If $\theta \in \Theta_0$ but we incorrectly decide to reject the null
Type II Error: If $\theta \in \Theta_0^c$ but we incorrectly fail to reject the null

Probability of Type I error: $P_\theta(\underline{X} \in R)$ when $\theta \in \Theta_0$
Probability of Type II error: $P_\theta(X \in R^c)$ when $\theta \in \Theta_0^c$
$= 1 - P_\theta(X \in R)$

		Accept H_1	Reject H_0
Truth	H_0	correct	Type I error
	H_1	Type II error	correct

Definition 8.3.1 The power function of a hypothesis test w/ rejection region R is the function of θ defined by $\beta(\theta) = P_\theta(\underline{X} \in R)$

Example 8.3.2 $X \sim \text{binomial}(5,\theta)$

$H_0: \theta \le \frac{1}{2}$

$H_1: \theta > \frac{1}{2}$

* Consider the test that Rejects Ho iff $X=5$

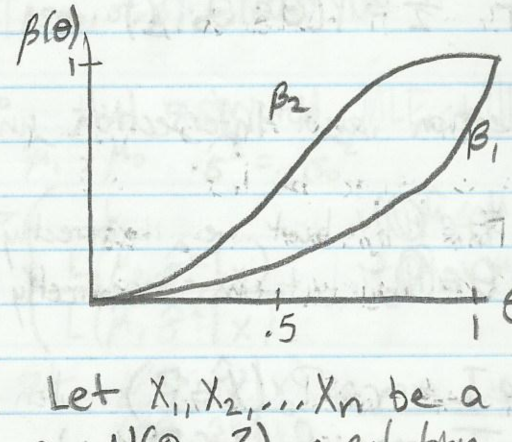
Power function $= \beta_1(\theta) = P_\theta(X \in R) = P(X=5) = \theta^5$

- Prob(Type I error): $\beta_1(\theta) \le (\frac{1}{2})^5 = .0312$
- Prob(Type II error): $\beta_1(\theta)$ is too small for $\theta > \frac{1}{2}$ so the Prob of type II error is too high

* Consider the test that Rejects H_0 iff $X = 3,4,5$

Power function $= \beta_2(\theta) = P_\theta(X \in R) = P(X=3,4,5)$

$$= \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{4}\theta^4(1-\theta)^1 + \binom{5}{5}\theta^5(1-\theta)^0$$



Example 8.3.3 Let $X_1, X_2, \ldots X_n$ be a random sample from a $N(\theta, \sigma^2)$ population w/ known σ^2

$H_0: \theta \le \theta_0$

$H_1: \theta > \theta_0$

Likelihood Ratio Test statistic yields Reject Ho if

$$\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} > c$$

$$\beta(\theta) = P_\theta\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} > c\right) = P\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)$$

$$= P_\theta\left(Z > c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)$$

Note: $\beta(\theta)$ is an increasing function of θ

$\lim_{\theta\to-\infty}\beta(\theta) = 0$ & $\lim_{\theta\to\infty}\beta(\theta) = 1$ & $\beta(\theta_0) = \alpha$ if $P(Z > c') = \alpha$

Suppose the experimenter wishes to have a maximum type I error probability of .1, and maximum type II error probability of .2 if $\theta \geq \theta_0 + \sigma$.

* We need to choose c and n to acheive this *

- Reject $H_0: \theta \leq \theta_0$ if $\dfrac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c$
- $\beta(\theta) = P\left(Z > c + \dfrac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$
- Because $\beta(\theta)$ is an increasing function of θ it suffices to
- show $\beta(\theta_0) = .1$ and $\beta(\theta_0 + \sigma) = .8$
- $c = 1.28$ we acheive $\beta(\theta_0) = P(Z > 1.28) = .1$
- choose $n \ni \beta(\theta + \sigma^2) = P(Z > 1.28 - \sqrt{n}) = .8 \rightarrow n = 4.49 \rightarrow n = 5$

Definition 8.3.5 For $0 \leq \alpha \leq 1$, a test w/ power function $\beta(\theta)$ is a size α test if $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$

Definition 8.3.6 For $0 \leq \alpha \leq 1$, a test w/ power function $\beta(\theta)$ is a level α test if $\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$

Definition 8.3.9 A test with power function $\beta(\theta)$ is unbiased if $\beta(\theta') \geq \beta(\theta'')$ $\forall$ $\theta' \in \Theta_0^c$ and $\theta'' \in \Theta_0$

Example 8.3.10 • A LRT of $H_0: \theta \leq \theta_0$

$H_1: \theta > \theta_0$

has $\beta(\theta) = P\left(Z > c + \dfrac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$ where $Z \sim N(0,1)$

- As $\beta(\theta)$ is an increasing function of θ

$\beta(\theta) > \beta(\theta_0) = \max_{t \leq \theta_0} \beta(t)$ $\forall$ $\theta > \theta_0$

$\therefore$ this test is unbiased

Definition 8.3.11: Let $\mathcal{C}$ be a class of tests for $H_0: \theta \in \Theta_0$
$H_1: \theta \in \Theta_0^c$

- A test in $\mathcal{C}$ is uniformly more powerful (UMP) if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$
- * We consider $\mathcal{C}$ = class of all level α tests

Theorem 8.3.12 Consider testing $H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$

w/ Rejection Region $R \ni$

① $\underline{x} \in R$ if $f(\underline{x}|\theta_1) > k f(\underline{x}|\theta_0)$
$\underline{x} \in R^c$ if $f(\underline{x}|\theta_1) < k f(\underline{x}|\theta_0)$ } for some $k \geq 0$

and ② $\alpha = P_{\theta_0}(\underline{X} \in R)$

Then

a) Any test satisfying ① and ② is a UMP level α test

b) If $\exists$ a test satisfying ① and ② w/ $k > 0$ then every UMP level α test is a size α test satisfying ② and every UMP level α test satisfies ① except on a set A satisfying $P_{\theta_0}(\underline{X} \in A) = P_{\theta_1}(\underline{X} \in A) = 0$

Proof 389

Corollary 8.3.13
- Consider testing $H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$
- Suppose $T(\underline{X})$ is a sufficient statistic for θ and $g(t|\theta_i)$ is the pdf or pmf of T corresponding to θ_i, $i = 0, 1$.

Then any test based on T w/ rejection region S is a UMP level α test if

① $t \in S$ if $g(t|\theta_1) > k g(t|\theta_0)$
② $t \in S^c$ if $g(t|\theta_1) < k g(t|\theta_0)$ } for some $k \geq 0$

where

$\alpha = P_{\theta_0}(T \in S)$

proof p 390

Example 8.3.14 $X \sim \text{Binomial}(2, \theta)$ We want to test

$H_0: \theta = \frac{1}{2}$
$H_1: \theta = \frac{3}{4}$

$$\frac{f(0 \mid \theta = \frac{3}{4})}{f(0 \mid \theta = \frac{1}{2})} = \frac{1}{4}$$

$$\frac{f(1 \mid \theta = \frac{3}{4})}{f(1 \mid \theta = \frac{1}{2})} = \frac{3}{4}$$

$$\frac{f(2 \mid \theta = \frac{3}{4})}{f(2 \mid \theta = \frac{1}{2})} = \frac{9}{4}$$

1. If we choose $\frac{3}{4} < k < \frac{9}{4}$ the Neyman-Pearson Lemma says that the test that rejects H_0 if $X = 2$ is the UMP level α test w/ $\alpha = P(X = 2 \mid \theta = \frac{1}{2}) = \frac{1}{4}$
2. for $\frac{1}{4} < k < 3/4$ reject H_0 if $X = 1$ or 2 is the UMP level α test w/ $\alpha = P(X = 1 \text{ or } 2 \mid \theta = \frac{1}{2}) = 3/4$

Example 8.3.15 • $X_1, X_2, \ldots X_n$ is a random sample from $N(\theta, \sigma^2)$ w/ σ^2 known.

• $\bar{x}$ is a sufficient statistic for θ

• Consider $\left.\begin{array}{l} H_0: \theta = \theta_0 \\ H_1: \theta = \theta_1 \end{array}\right\} \ni \theta_0 > \theta_1$

$$g(\bar{x} \mid \theta_1) > k\, g(\bar{x} \mid \theta_0) \equiv \bar{x} < \underbrace{\left[\frac{\frac{(2\sigma^2 \log(k))}{n} - \theta_0^2 + \theta_1^2}{2(\theta_1 - \theta_0)} \right]}$$

increases from $-\infty$ to ∞ as k increases from 0 to ∞

$\therefore$ By 8.3.13 the test $\bar{x} < c$ is the UMP level α test where $\alpha = P_{\theta_0}(\bar{x} < c)$

*What if we don't have a simple hypothesis?

<u>Definition 8.3.16</u> A family of pdfs or pmfs $\{g(t|\theta) \mid \theta \in \Theta\}$ for a univariate RV T w/ $\theta \in \mathbb{R}$ has a monotone likelihood Ratio (MLR) if $\forall\, \theta_2 > \theta_1 \rightarrow \frac{g(t|\theta_2)}{g(t|\theta_1)}$ is a monotone increasing function of t on $\{t \mid g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\}$

have MLR $\left\{\begin{array}{l} \text{* · Normal w/ known variance \& unknown mean} \\ \text{· Poisson} \\ \text{· Binomial} \\ \text{· Any exponential family w/ } w(\theta) \text{ nondecreasing} \end{array}\right.$

<u>Theorem 8.3.17 Karlin-Rubin</u>; · Consider testing $H_0: \theta \le \theta_0$
$H_1: \theta > \theta_0$

· Suppose T is a sufficient statistic of θ and the family $\{g(t|\theta) \mid \theta \in \Theta\}$ of T has MLR

· <u>Then</u> $\forall\, t_0$ the test rejects H_0 iff $T > t_0$ is a UMP level α test where $\alpha = P_{\theta_0}(T > t_0)$

<u>proof</u> p 391

<u>Example 8.3.18</u> (continued 8.3.15) · Consider $H_0: \theta \ge \theta_0$
$H_1: \theta < \theta_0$

Reject if $\bar{X} < -\frac{\sigma Z_\alpha}{\sqrt{n}} + \theta_0$

· As $\bar{X}$ is sufficient and its distribution has MLR the test is a UMP level α test

· As $\beta(\theta) = P\left(\bar{X} < -\frac{\sigma Z_\alpha}{\sqrt{n}} + \theta_0\right)$ is a decreasing function of θ

$\alpha = \sup_{\theta \ge \theta_0} \beta(\theta) = \beta(\theta_0)$

<u>Note</u> $X_1, X_2, \dots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ w/ σ^2 known

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

Does not have a UMP test

* When no UMP test exists within $\mathcal{C}$ we might try to find a UMP level α test within the class of unbiased tests
* For the above example

Reject: $\bar{x} > \dfrac{\sigma z_{\alpha/2}}{\sqrt{n}} + \theta_0$ or $\bar{x} < -\dfrac{\sigma z_{\alpha/2}}{\sqrt{n}} + \theta_0$

is a UMP unbiased level α test

* Although we can find tests w/ higher power at certain values of θ they would be unbiased and, thus, not applicable to our statement as we are only considering unbiased tests.