Chapter 10

10.1.1 Consistency - the estimator converges to the correct value as the sample size becomes infinite

$\lim_{n\to\infty} P_\theta(|W_n-\theta|<\epsilon)=1$ for $X_1, X_2, \ldots \sim f(x|\theta)$ w/ estimators $W_n = W_n(X_1, \ldots X_n)$

* Think convergence in prob *

Example 10.1.2 Let $X_1, X_2 \ldots$ be iid $N(\theta,1)$ and consider the seq $\bar{X}_n = \frac{1}{n}\sum_1^n x_i$

$\bar{X} \sim N(\theta, 1/n)$

$$P(|\bar{X}_n-\theta|<\epsilon) = \underbrace{\int_{\theta-\epsilon}^{\theta+\epsilon} \left(\frac{n}{2\pi}\right)^{1/2} e^{-(n/2)(\bar{x}_n-\theta)^2}}_{Normal} d\bar{x}_n$$

$$\int_{-\epsilon}^{\epsilon} \left(\frac{n}{2\pi}\right)^{1/2} e^{-(n/2)y^2} dy \qquad y = \bar{x}_n - \theta$$

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \left(\frac{1}{2\pi}\right)^{1/2} e^{-(1/2)t^2} dy \qquad t = y\sqrt{n}$$

$$P(-\epsilon\sqrt{n} \le Z \le \epsilon\sqrt{n}) \qquad Z \sim N(0,1)$$

$\to 1$ as $n \to \infty$

$\therefore \bar{X}$ is a consistant estimator of θ

Theorem 10.1.3 If W_n is a sequence of estimators of θ satisfying

i: $\lim_{n\to\infty} Var_\theta(W_n)=0$

ii: $\lim_{n\to\infty} Bias(W_n)=0$

$\forall\, \theta \in \Theta$, then W_n is a consistent sequence of estimators

Example 10.1.4 $E_\theta(\bar{X}_n)=\theta$ and $Var(\bar{X}_n)=\frac{1}{n}$

i: $\lim_{n\to\infty} \frac{1}{n} = 0$

ii: $\lim_{n\to\infty} \theta-\theta = 0$

$\therefore W_n = \bar{X}_n$ is a consistent sequence of estimators

Theorem 10.1.5
- Let W_n be a consistent sequence of estimators of θ
- Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of constants $\ni$

  i: $\lim_{n\to\infty} a_n = 1$

  ii: $\lim_{n\to\infty} b_n = 0$
- Then $U_n = a_n W_n + b_n$ is a consistent sequence of estimators

Theorem 10.1.6 Consistency of MLE: Let $X_1, X_2, \dots \overset{iid}{\sim} f(x|\theta)$ and let $L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta)$. Let $\hat{\theta}$ denote the MLE. Let $\tau(\theta)$ be a continuous function of θ.
Then, under regularity conditions:

$$\lim_{n\to\infty} P_\theta(|\tau(\hat{\theta}) - \tau(\theta)| \geq \epsilon) = 0$$

ie $\tau(\hat{\theta})$ is a consistent estimator of $\tau(\theta)$

Definition 10.1.7 For an estimator T_n, if $\lim_{n\to\infty} k_n \mathrm{Var}(T_n) = \tau^2 < \infty$ where $\{k_n\}$ is a sequence of constants, then τ^2 is called the limiting variance

I think we skipped 10.1.2 - 10.3

Theorem 10.3.1
- For $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$
- Suppose $X_1, X_2, \dots X_n \overset{iid}{\sim} f(x|\theta)$, $\hat{\theta}$ is the MLE and $f(x|\theta)$ satisfies regularity conditions

Then under H_0 as $n \to \infty$ $\quad -2\log(\lambda(\mathbf{X})) \xrightarrow{d} \chi^2_1$

Example 10.3.2
$H_0: \lambda = \lambda_0$
$H_a: \lambda \neq \lambda_0$
$X_1, X_2, \dots X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$
$-2\log(\lambda(\mathbf{x})) = 2\left[(\lambda_0 - \hat{\lambda}) - \hat{\lambda}\log(\lambda_0/\hat{\lambda})\right]$ NOTE $\hat{\lambda} = \Sigma x_i/n$
Reject H_0 if $-2\log(\lambda(\mathbf{x})) > \chi^2_{1,\alpha}$

Theorem 10.3.3 Let $X_1, X_2, \dots X_n \sim f(x|\theta)$, under regularity conditions, if $\theta \in \Theta_0$ then $-2\log(\lambda(\mathbf{X})) \xrightarrow{d} \chi^2$ as $n \to \infty$ and the degrees of freedom is size Θ_0 − size $\Theta = \nu$
$H_0: \theta \in \Theta_0$
reject iff $-2\log\lambda(\mathbf{x}) \geq \chi^2_{r,\alpha}$
Prb(Type I error) $\approx \alpha$ if $\theta \in \Theta_0$ and n is large

$$\lim_{n\to\infty} P_\theta(\text{reject } H_0) = \alpha \quad \forall\, \theta \in \Theta_0$$

Example 10.3.4 · Let $\Theta = (p_1, p_2, p_3, p_4, p_5)$ where p_j's are nonnegative and $\Sigma p_j = 1$

· Suppose $X_1, X_2, \dots, X_n$ are iid discrete RV and $P_\theta(X_i = j) = p_j$ for $j = 1, 2, \dots 5$

· Thus $f(j|\theta) = p_j$ is the pmf of X_i and

$$L(\theta|\underline{x}) = \prod_i^n f(x_i|\theta) = p_1^{y_1} p_2^{y_2} p_3^{y_3} p_4^{y_4} p_5^{y_5} \qquad y_j = \#\ x_i = j$$

$H_0: p_1 = p_2 = p_3$ and $p_4 = p_5$

H_a: not H_0

$v =$ size Θ - size Θ_0 $= 4 - 1$

$\vdots$

$$-2\log(\lambda(x)) = 2\sum_i^5 y_i \log\left(\frac{y_i}{m_i}\right) \qquad m_1 = m_2 = m_3 = \left(\frac{y_1 + y_2 + y_3}{3}\right)$$

$$m_4 = m_5 = \left(\frac{y_4 + y_5}{2}\right)$$

Reject H_0 if $-2\log(\lambda(x)) \geq \chi^2_{3,\alpha}$

Wald Test $\quad Z_n = \dfrac{W_n - \theta_0}{S_n}$

θ_0 = the hypothesized value of θ

W_n = an estimator of θ

S_n = st. error for θ

$$= \frac{1}{\sqrt{I_n(W_n)}} \approx \frac{1}{\sqrt{\frac{\partial^2}{\partial\theta^2}\log(L(\theta|x))\big|_{\theta = W_n}}} \quad \text{(information \#)}$$

Example 10.3.5 · Let $X_1, \dots, X_n \sim$ Bernoulli(p)

· $H_0: p \leq p_0$ vs $H_a: p > p_0$ where $0 < p_0 < 1$

· The MLE of p is $\hat{p} = \sum_i^n \frac{X_i}{n}$

· CLT $\frac{\hat{p}_n - p}{\sigma_n} \longrightarrow N(0,1)$ where $\sigma_n = \sqrt{p(1-p)/n}$

$S_n = \sqrt{\hat{p}(1-\hat{p})/n}$

and $\sigma_n / S_n \longrightarrow 1$

$$\therefore \frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \longrightarrow N(0,1)$$

WALD: $$\frac{\hat{p}_n - p_0}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \longrightarrow n(0,1)$$

Reject H_0 if $Z_n > Z_\alpha$

For $H_a: p \neq p_0$

Reject H_0 if $|Z_n| > Z_{\alpha/2}$

Score Statistic $S(\theta) = \frac{\partial}{\partial\theta} \log(f(\underline{x}|\theta)) = \frac{\partial}{\partial\theta} \log(L(\theta|\underline{x}))$

$\forall\, \theta \quad E_\theta(S(\theta)) = 0$

$$var(S(\theta)) = E_\theta\left(\left(\frac{\partial}{\partial\theta}\log(L(\theta|\underline{x}))\right)^2\right) = -E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log(L(\theta|\underline{x}))\right)$$

$$= I_n(\theta)$$

Score Test $Z_S = \frac{S(\theta_0)}{\sqrt{I_n(\theta_0)}}$

Reject H_0 if $|Z_S| > z_{\alpha/2}$

Example 10.3.6 $H_0: p = p_0$

$H_a: p \neq p_0$

$$S(p) = \frac{\hat{p}_n - p}{p(1-p)/n} \longrightarrow I_n(p) = \frac{n}{p(1-p)}$$

Thus $Z_S = \frac{\hat{p}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$