Chapter 25 (assuming balance ie: $n_i = n$)

What if our factors are random variables? ie choose x out of y levels.

Random Cell Means Model $Y_{ij} = \mu_i + \epsilon_{ij}$

$\uparrow$ no longer constant, now $\sim N(\mu_\cdot, \sigma_\mu^2)$

$$E(Y_{ij}) = \mu_\cdot$$

$$\sigma^2(Y_{ij}) = \sigma_Y^2 = \sigma_\mu^2 + \sigma^2$$

$$Y_{ij} \sim N(\mu_\cdot, \sigma_Y^2)$$

$$\sigma(Y_{ij}, Y_{ij'}) = \sigma_\mu^2 \quad j \neq j'$$

$$\sigma(Y_{ij}, Y_{i'j'}) = 0 \quad i = i'$$

$$\rho(Y_{ij}, Y_{ij'}) = \frac{\sigma_\mu^2}{\sigma_Y^2} = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma^2}$$

Test $H_0: \sigma_\mu^2 = 0$

$H_a: \sigma_\mu^2 > 0$

$F^* = MSTreatment / MSE$

reject if $F^* > F[1-\alpha, r-1, r(n-1)]$

Estimation of $\mu_\cdot$ $\hat{\mu}_\cdot = \bar{Y}_{..}$

$$\sigma^2(\bar{Y}_{..}) = \frac{n\sigma_\mu^2 + \sigma^2}{rn}$$

$$s^2(\bar{Y}_{..}) = MSTreatment / r \cdot n$$

Confidence interval for $\mu_\cdot$ $\bar{Y}_{..} \pm t(1-\alpha/2; r-1)\, s(\bar{Y}_{..})$

Interval Estimate of $\frac{\sigma_\mu^2}{(\sigma_\mu^2 + \sigma^2)}$

$$L = \frac{1}{n}\left[\frac{MSTreatment}{MSE}\left(\frac{1}{F[1-\alpha/2, r-1, r(n-1)]}\right) - 1\right]$$

$$U = \frac{1}{n}\left[\frac{MSTreatment}{MSE}\left(\frac{1}{F[\alpha/2, r-1, r(n-1)]}\right) - 1\right]$$

$$\left(\frac{L}{1+L}, \frac{U}{1+U}\right)$$

Interval Estimate of σ^2: $\left(\dfrac{r(n-1)MSE}{\chi^2[1-\alpha/2, r(n-1)]}, \dfrac{r(n-1)MSE}{\chi^2[\alpha/2, r(n-1)]}\right)$

Point estimator of σ_μ^2 $\quad s_\mu^2 = \dfrac{MSTR - MSE}{n}$

Interval estimator of σ_μ^2 * See Satterwaite Procedure 25.24 - p 1043
* See MLS Procedure 25.32 - p 1045

Random Effects Model: $\mu_i = \mu_\cdot + \tau_i$ where $\tau_i = \mu_i - \mu_\cdot$

$Y_{ij} = \mu_\cdot + \tau_i + \epsilon_{ij}$ (single-factor random cell)

Two factors Model II: $Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$

- $\mu_{\cdot\cdot}$ constant
- $\alpha_i, \beta_j, \alpha\beta_{ij}$ are $\perp$ w/ exp. value $\sigma_\alpha^2, \sigma_\beta^2, \sigma_{\alpha\beta}^2$
- $\epsilon_{ij} \sim N(0, \sigma^2)$

$E(Y_{ijk}) = \mu_{\cdot\cdot}$

$\sigma^2(Y_{ijk}) = \sigma_Y^2 = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta} + \sigma^2$

$\sigma(Y_{ijk}, Y_{ij'k'}) = \sigma_\alpha^2$

$\sigma(Y_{ijk}, Y_{i'jk'}) = \sigma_\beta^2$

$\sigma(Y_{ijk}, Y_{ijk'}) = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta}^2$

$\sigma(Y_{ijk}, Y_{i'j'k'}) = 0$

Mixed factors Model III: $Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$

- $\mu_{\cdot\cdot}$ constant
- α_i constants $\ni \Sigma\alpha_i = 0$
- $\beta_j \sim N(0, \sigma_\beta^2)$
- $(\alpha\beta)_{ij} \sim N(0, \frac{a-1}{a}\sigma_{\alpha\beta}^2)$
  - $\sum_i (\alpha\beta)_{ij} = 0 \; \forall j$

$E(Y_{ijk}) = \mu_{\cdot\cdot} + \alpha_i$

$\sigma^2(Y_{ijk}) = \sigma_Y^2 = \sigma_\beta^2 + \frac{a-1}{a}\sigma_{\alpha\beta}^2 + \sigma^2$

$\sigma(Y_{ijk}, Y_{ijk'}) = \sigma_\beta + \frac{a-1}{a}\sigma_{\alpha\beta}^2$

$\sigma(Y_{ijk}, Y_{i'jk'}) = \sigma_\beta^2 - \frac{1}{a}\sigma_{\alpha\beta}^2$

$\sigma(Y_{ijk}, Y_{i'j'k'}) = 0$

Mean Squares & Tests for effects Table 25.5 and 25.6 - p 1052

example: $H_o: \sigma_\alpha^2 = 0$

$H_a: \sigma_\alpha^2 > 0$

$F^* = \dfrac{MSA}{MSAB}$

Reject if $F^* > F[1-\alpha, a-1, (a-1)(b-1)]$

Estimation of σ_β^2 $\quad s_\beta^2 = \frac{MSB - MSE}{na}$

Interval estimate of σ_β^2 see 25.51 - p 1055

Estimation of fixed effects in mixed model

$\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}$

$\hat{\mu}_i = \bar{Y}_{...} + (\bar{Y}_{i..} - \bar{Y}_{...}) = \bar{Y}_{i..}$

$L = \sum c_i \alpha_i$ where $\sum c_i = 0 \rightarrow \hat{L} = \sum c_i \bar{Y}_{i..}$

Variance of Estimators

$s^2(\hat{\alpha}_i) = \frac{MSAB}{bn}$ $\quad s^2(\hat{\mu}_{i.}) = \frac{a-1}{nab} MSAB + \frac{1}{nab} MSB$

$s^2(\hat{L}) = \frac{MSAB}{bn} \sum c_i^2$

Confidence interval for L $\quad \hat{L} \pm t[1-\alpha/2, (a-1)(b-1)]s(\hat{L})$

for multiple comp. replace t with $T = \frac{1}{\sqrt{2}} q[1-\alpha, a, (a-1)(b-1)]$

Confidence Interval for $\hat{\mu}_{i.}$ $\quad \hat{\mu}_{i.} \pm t(1-\alpha/2, df)s(\mu_{i.})$

(df) defined in 25.65 p 1059

Additive RCBD w/ Random Block Effects $\quad Y_{ij} = \mu_{..} + \underbrace{\rho_i}_{\text{random factor}} + \underbrace{\tau_j}_{\text{treatment}} + \epsilon_{ij}$

$E(Y_{ij}) = \mu_{..} + \tau_j$ $\quad$ $\mu_{..}$ is constant

$\sigma^2(Y_{ij}) = \sigma_Y^2 = \sigma_\rho^2 + \sigma^2$ $\quad$ ρ_i are ind $N(0, \sigma_\rho^2)$

$\sigma^2(Y_{ij}, Y_{ij'}) = \sigma_\rho^2$ $\quad$ τ_j constants $\ni \sum \tau_j = 0$

$\sigma(Y_{ij}, Y_{i'j'}) = 0$ $\quad$ ϵ_{ij} are ind $N(0, \sigma^2) \perp \rho_i$

$\omega = \frac{\sigma_\rho^2}{\sigma_Y^2}$ = correlation between any two observations from same block

$$\sigma^2(\underline{Y}) = \sigma_Y^2 \begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{bmatrix}$$

Sum of squares and degrees of freedom : Table 25.8 p 1063

Interaction RCBD w/ random block effects

$Y_{ij} = \mu_{..} + \rho_i + \tau_j + (\rho\tau)_{ij} + \epsilon_{ij}$

- $\mu_{..}$ is a constant
- ρ_i are indep. $N(0, \sigma_\rho^2)$
- τ_j constants $\ni \sum \tau_j = 0$
- $(\rho\tau)_{ij} \sim N(0, \frac{r-1}{r}\sigma_{\rho\tau}^2)$
  $\ni \sum_j (\rho\tau)_{ij} = 0 \ \forall i$
  w/ $\sigma((\rho\tau)_{ij}, (\rho\tau)_{ij'}) = -\frac{1}{r}\sigma_{\rho\tau}^2$
- $(\rho\tau)_{ij} \perp\!\!\!\perp \rho_i$
- $\epsilon_{ij} \sim N(0, \sigma^2) \perp\!\!\!\perp \rho_i$ and $(\rho\tau)_{ij}$

$E(Y_{ij}) = \mu_{..} + \tau_j$

$\sigma^2(Y_{ij}) = \sigma_Y^2 = \sigma_\rho^2 + \frac{r-1}{r}\sigma_{\rho\tau}^2 + \sigma^2$

$\sigma(Y_{ij}, Y_{ij'}) = \sigma_\rho^2 - \frac{1}{r}\sigma_{\rho\tau}^2$

$\sigma(Y_{ij}, Y_{i'j'}) = 0$

$$\omega = \frac{\sigma_\rho^2 - \frac{1}{r}\sigma_{\rho\tau}^2}{\sigma_Y^2}$$

Sum of squares and degrees of freedom Table 25.8 p 1063

Three Factor Random Factor Effects Model II

- $Y_{ijkm} = \mu_{...} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkm}$
- $\mu_{...}$ is constant
- $\alpha_i, \beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}, \epsilon_{ijk}$ are independent normal RV w/ variances $\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2, \sigma_{\alpha\beta}^2, \sigma_{\alpha\gamma}^2, \sigma_{\beta\gamma}^2, \sigma_{\alpha\beta\gamma}^2, \sigma^2$
- $E(Y_{ijkm}) = \mu_{...}$
- $\sigma^2(Y_{ijkm}) = \sigma_Y^2 = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\gamma^2 + \sigma_{\alpha\beta}^2 + \sigma_{\beta\gamma}^2 + \sigma_{\alpha\beta\gamma}^2 + \sigma^2$

Three Factor Mixed Factor Effects Model III

- $Y_{ijkm} = \mu_{..} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkm}$
- $\mu_{...}$ is a constant
- α_i are constants
- $\beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ are pairwise ind normal RW w expectations zero and constant variance
- $\epsilon_{ijkm} \sim N(0, \sigma^2)$
- $\sum_i \alpha_i = \sum_i (\alpha\beta)_{ij} = \sum_i (\alpha\gamma)_{ik} = \sum_i (\alpha\beta\gamma)_{ijk} = 0$
- $E(Y_{ijkm}) = \mu_{...} + \alpha_i$

Exact F Test: Test for BC interactions $F^* = \frac{MSBC}{MSE}$
Test for A main effects $F^* = \frac{MSA}{MSAB}$ (if $\sigma_{\alpha\gamma} = 0$)

Satterthwaite Approx. F Test when you don't know what interactions are zero

$$H_0: \sigma_\alpha^2 = 0$$
$$H_1: \sigma_\alpha^2 \neq 0$$
$$F^* = \frac{MSA}{MSAB + MSAC - MSABC}$$

skipped 25.7

SAS
```
/*spaghetti plot*/
proc sgplot nolegend;
   series x=A y=Y / group=B;
   scatter x=A y=Y / group=B markerchar=B;
run;
/*Way 1: glm*/
proc glm plots=all;
   class A B;
   model Y= A B;
   random B;
   lsmeans A / pdiff adjust=Tukey alpha=.05 cl;
run;
/*way 2: glimmix-better*/ ods output SolutionR=rand; /*for σ̂p*/
proc glimmix plots=all;
   class A B;
   model Y= A / s chisq;
   random B;
   lsmeans A / pdiff adjust=tukey alpha=.05;
   covtest zerog; /*test σp=0*/
run;
```

```
proc print data=rand;
run;
proc univariate data=rand normal;
    var estimate;
run;
```

/* check normality of ρ_i estimates */