Ch. 4 Skipped

Ch. 5 The matrix approach

- Square matrix: same number of rows and columns
- Row vector: one row, several columns
- Column vector: one column, several rows

- Transpose

$$\underset{3\times 2}{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}^T} = \underset{2\times 3}{\begin{bmatrix} a_1 & a_3 & a_5 \\ a_2 & a_4 & a_6 \end{bmatrix}}$$

- Addition/Subtraction

$$\overset{3\times 2}{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}} + \overset{3\times 2}{\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}} = \overset{3\times 2}{\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{bmatrix}}$$

- Scalar multiplication

$$a \overset{3\times 2}{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}} = \begin{bmatrix} aa_1 & aa_2 \\ aa_3 & aa_4 \\ aa_5 & aa_6 \end{bmatrix}$$

- Matrix multiplication

$$\underset{2\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}} * \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} a_1b_1 + a_2b_2 + a_3b_3 \\ a_4b_1 + a_5b_2 + a_6b_3 \end{bmatrix}}$$

$2\times 3 = 3\times 1$

· Symmetric matrix if $A = A^T$

ie $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix}$

· Diagonal matrix all off-diagonal elements = 0

ie $A \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

· Identity Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Note $IA = A = AI$

· Inverse of a 2x2

Note determinat = $a_1 a_4 - a_2 a_3$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1} = \frac{1}{[a_1 a_4 - a_2 a_3]} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$$

· Inverse of a diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

Assorted Matrix properties

1) $A+B=B+A$
2) $(A+B)+C = A+(B+C)$
3) $(AB)C = A(BC)$
4) $C(A+B) = CA+CB$
5) $K(A+B) = KA+KB$
6) $(A^T)^T = A$
7) $(A+B)^T = A^T+B^T$
8) $(AB)^T = B^T A^T$
9) $(ABC)^T = C^T B^T A^T$
10) $(AB)^{-1} = B^{-1}A^{-1}$
11) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
12) $(A^{-1})^{-1} = A$
13) $(A^T)^{-1} = (A^{-1})^T$

Expected Value of a Matrix

$$E(Y) = E\left(\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}\right) = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ E(Y_3) \end{bmatrix}$$

Variance-Covariance of a Matrix

$$\sigma^2(Y) = \sigma^2\left(\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}\right) = \begin{bmatrix} \sigma^2(Y_1) & \sigma^2(Y_1, Y_2) & \sigma^2(Y_1, Y_3) \\ \sigma^2(Y_2, Y_1) & \sigma^2(Y_2) & \sigma^2(Y_2, Y_3) \\ \sigma^2(Y_3, Y_1) & \sigma^2(Y_3, Y_2) & \sigma^2(Y_3) \end{bmatrix}$$

Multivariate Normal

$Y = [Y_1, Y_2, \dots Y_p]'$ $\quad \mu = [\mu_1, \mu_2, \dots \mu_p]'$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix}$$

$$f(Y) = \frac{1}{2\pi^{p/2} |\Sigma|^{1/2}} e^{\left[-\frac{1}{2}(Y-\mu)' \Sigma^{-1} (Y-\mu)\right]}$$

Linear Regression in Matrix Terms

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i \rightarrow \begin{array}{l} y_1 = \beta_0 + \beta_1 x_1 + \epsilon_1 \\ y_2 = \beta_0 + \beta_1 x_2 + \epsilon_2 \\ \vdots \\ y_n = \beta_0 + \beta_1 x_n + \epsilon_n \end{array}$$

Let, $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$ $\underline{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$ $\underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ $\underline{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$

$$\underline{Y} = \underline{\beta}\underline{X} + \epsilon$$

NOTE: $E(\underline{Y}) = \underline{\beta}\underline{X}$

$E(\underline{\epsilon}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \sigma^2(\underline{\epsilon}) = \sigma^2 \underline{I}$

Normal equations in matrix terms

$$\underline{X}'\underline{X}\,\underline{b} = \underline{X}'\underline{Y} \Rightarrow \begin{array}{l} nb_0 + b_1 \Sigma X_i = \Sigma Y_i \\ b_0 \Sigma X_i + b_1 \Sigma X_i^2 = \Sigma X_i Y_i \end{array}$$

Coefficent estimators in matrix terms

$$\underline{b} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Y}$$

Fitted Values in matrix terms

$$\underline{\hat{Y}} = \underline{X}\underline{b} = \underline{X}(\underline{X}'X)^{-1}\underline{X}'\underline{Y} = \underline{H}\underline{Y}$$

Hat Matrix

$H = \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}\underline{Y}$

Note: Matrix H is symmetric

$\underline{H}\underline{H} = \underline{H}$

Residuals in matrix terms

$\underline{e} = \underline{Y} - \underline{\hat{Y}} = \underline{Y} - \underline{X}\underline{b} = (\underline{I} - \underline{H})\underline{Y}$

$\sigma^2(e) = \sigma^2(I - H)$

$s^2(e) = MSE(I - H)$

ZD

Sum of Squares in matrix terms

$SSTO = \underline{Y}'\underline{Y} - (\frac{1}{n})\underline{Y}'\underline{J}\underline{Y}$ where $J = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$

$SSE = \underline{Y}'\underline{Y} - \underline{b}'\underline{X}'\underline{Y}$

$SSR = \underline{b}'\underline{X}'\underline{Y} - (\frac{1}{n})\underline{Y}'\underline{J}\underline{Y}$

Mean Squares in matrix terms

$MSR = \frac{SSR}{p-1}$

$MSE = \frac{SSE}{n-p}$