1. Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$. Suppose that the k parametric functions $\boldsymbol{\lambda}_1'\boldsymbol{\beta}, \boldsymbol{\lambda}_2'\boldsymbol{\beta}, ..., \boldsymbol{\lambda}_k'\boldsymbol{\beta}$ are each estimable. Show that $\sum_{i=1}^k d_i \boldsymbol{\lambda}_i'\boldsymbol{\beta}$, where $d_i \in \mathcal{R}$, is also estimable.

2. Consider the fixed-effects nested model

$$Y_{ijk} = \mu + \alpha_i + \beta_{ij} + \epsilon_{ijk},$$

where $E(\epsilon_{ijk}) = 0$, for $i = 1, 2$, $j = 1, 2, 3$, and $k = 1, 2$.
(a) Write this model in $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ form.
(b) Show that $\alpha_1 - \alpha_2$ is not estimable.
(c) Show that $\beta_{11} - \beta_{12}$ is estimable but that $\beta_{11} - \beta_{21}$ is not. Give an unbiased estimator of $\beta_{11} - \beta_{12}$.
(d) Show that μ, α_1, α_2 are jointly nonestimable.

3. Consider the ANCOVA model

$$Y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \epsilon_{ij},$$

where $E(\epsilon_{ijk}) = 0$, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Assume that $\sum_{j=1}^4 (x_{ij} - \overline{x}_{i+})^2 > 0$, for $i = 1, 2$.
(a) Write this model in $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ form.
(b) Find necessary and sufficient conditions for $\boldsymbol{\lambda}'\boldsymbol{\beta}$ to be estimable.
(c) Find a set of r linearly independent estimable functions.
(d) Find a maximal set of jointly nonestimable functions.

4. Let $\mathbf{Y}$ be a 6×1 vector,

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix},$$

and $\boldsymbol{\beta} = (\mu, \theta_1, \theta_2, \theta_3, \theta_4)'$. Assume the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ holds, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ denote the columns of $\mathbf{X}$. Note that $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 = \mathbf{0}$.
(a) Find the components of $E(\mathbf{Y})$ in terms of $\mu, \theta_1, \theta_2, \theta_3$, and θ_4.
(b) Show that $\mathbf{x}_0$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ are linearly independent.
(c) Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)'$. Give conditions on $\boldsymbol{\lambda}$, of the form $\boldsymbol{\lambda}'\mathbf{c}_i = 0$, $i = 1, 2, ..., s$, that are necessary and sufficient for $\boldsymbol{\lambda}'\boldsymbol{\beta}$ to be estimable. What is the value of $s = p - r$?
(d) Show that $\mu + \theta_3 - \theta_4$ is estimable.
(e) Give a nonestimable function of the form $\boldsymbol{\lambda}'\boldsymbol{\beta}$. Explain your answer.
(f) Explain briefly how you could use your answer in part (e) to force a particular solution

to the normal equations $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{Y}$. If you did this, would your nonestimable $\boldsymbol{\lambda}'\boldsymbol{\beta}$ in part (e) become estimable? Explain.

5. Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $E(\boldsymbol{\epsilon}) = \mathbf{0}$. Show that the following four statements are equivalent.

1. $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is estimable.
2. $\boldsymbol{\lambda}'\mathbf{a} = 0$, for every $\mathbf{a}$ such that $\mathbf{X}\mathbf{a} = \mathbf{0}$.
3. $\boldsymbol{\lambda} \in \mathcal{C}(\mathbf{X}'\mathbf{X})$.
4. $\boldsymbol{\lambda}' = \boldsymbol{\lambda}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X}$.

6. To detect the presence of harmful beetles in farm fields, experimenters placed six boards of each of four colors at random locations in a field of oats and measured the number of cereal leaf beetles trapped. The boards were covered with sticky material to trap the beetles easily. The data from the experiment are given below.

Treatment 1	Blue	16	8	20	24	14
Treatment 2	Green	33	32	20	25	37
Treatment 3	White	23	15	10	17	13
Treatment 4	Yellow	45	49	44	46	35

(a) Fit a one way fixed effects ANOVA model $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$ to these data. Report a least squares estimate of $\boldsymbol{\beta}$. Is this estimate unique?
(b) Fit a one way cell means ANOVA model $Y_{ij} = \mu_i + \epsilon_{ij}$ to these data. Report a least squares estimate of $\boldsymbol{\beta}$. Is this estimate unique?
(c) Write an estimable function that allows the experimenters to compare the average of the blue and green colors to the average of the white and yellow colors. Find the least squares estimate of this function using the estimate of $\boldsymbol{\beta}$ from both (a) and (b).
(d) Write a function of $\boldsymbol{\beta}$ that is not estimable, and show that the least squares "estimate" of it is different, depending on which estimate of $\boldsymbol{\beta}$ from (a) and (b) you use.
(e) Pick a maximal set of orthogonal contrasts and show that the sums of squares for the contrasts adds to the (corrected) model sum of squares.

7. Consider a linear model with $n = 3$ observations Y_1, Y_2, and Y_3 and four parameters μ, α_1, α_2, and α_3. Suppose that

$$\begin{aligned} E(Y_1) &= \mu + \alpha_1 + \alpha_2 + \alpha_3 \\ E(Y_2) &= \mu + \alpha_1 + \alpha_2 \\ E(Y_3) &= \mu + \alpha_1. \end{aligned}$$

Denote the parameter vector $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \alpha_3)'$.
(a) Write this model in $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ form.

(b) Characterize each of the following functions as estimable or nonestimable: μ, α_1, $\mu + \alpha_1$, $\mu + \alpha_1 + \alpha_2 + \alpha_3$. For each one, justify your answer.
(c) Write down the normal equations for this model.
(d) Find two solutions $\widehat{\boldsymbol{\beta}}_1$ and $\widehat{\boldsymbol{\beta}}_2$ to the normal equations.
(e) Find the least squares estimates of the estimable functions in part (b). Verify that these estimates are invariant to which of $\widehat{\boldsymbol{\beta}}_1$ and $\widehat{\boldsymbol{\beta}}_2$ is used.

8. This is a simple model involving spline functions. Data pairs (x_i, y_i) are collected. If $x_i < 0$, then $E(Y_i) = \mu$. If $x_i \geq 0$, then $E(Y_i) = \alpha + \beta x_i$. Consider the four data pairs $(-2, -3), (-1, -1), (1, 1), (2, 2)$.
(a) What are the vectors $\mathbf{Y}$ and $\boldsymbol{\beta}$ and the matrix $\mathbf{X}$ for this model?
(b) Compute a least squares estimate of $\boldsymbol{\beta}$. You can just write this down by considering what functions would minimize $Q(\boldsymbol{\beta})$.
(c) Now add the restriction that the two functions μ and $\alpha + \beta x$ must be equal at $x = 0$. Find a least squares estimate of $\boldsymbol{\beta}$ in this restricted model. The resulting function of x might be called a spline function.