STAT 535 Test 1 Practice Derivation Problems

1. Suppose we have n iid observations $Y_1, \ldots, Y_n$ that follow a gamma distribution with shape parameter known to be 3 and unknown rate parameter $\theta > 0$, i.e., having density

$$f(y|\theta) = \frac{\theta^3}{\Gamma(3)} y^{3-1} e^{-\theta y}, \quad y > 0.$$

The analyst wishes to perform inference about the unknown parameter $\theta > 0$.

(a) Write and simplify the likelihood function based on $Y_1, \ldots, Y_n$.

(b) Suppose the analyst chooses a gamma prior distribution with shape parameter s and rate parameter r (the pdf is given in notes). Explain why this choice is sensible, with respect to the support of θ.

(c) If the analyst believes, before examining the data, that θ has a mean of 3 and a standard deviation of 1, then what are suitable choices for the (hyper)parameters of the gamma prior distribution? Briefly explain.

(d) Under this likelihood and prior, derive the posterior distribution for θ, given $Y_1, \ldots, Y_n$.

(e) Give an expression for the posterior mean, and express this as weighted average of the MLE (which is $3/\bar{Y}$) and the prior mean.

2. (The Beta-Geometric model) Consider the following new Bayesian model, like in Exercise 5.19 but with n data observations rather than a single observation:

where the Geometric model has pmf $f(y|\theta) = \theta(1-\theta)^{y-1}$ for $y \in \{1, 2, \ldots\}$. Note that the mean of this geometric distribution is $1/\theta$.

(a) Write and simplify the likelihood function based on $Y_1, \ldots, Y_n$.

(b) Assume we choose a beta(α, β) prior for θ. Derive the posterior model for θ given observed data $y_1, \ldots, y_n$. If possible, identify the name of the posterior model and its parameters.

(c) Is the Beta model a conjugate prior for the Geometric data model?

(d) Write an expression for the posterior mean, which could be used as a Bayesian point estimate of θ.