9.5 The Rao-Blackwell Theorem and MVUEs

- This theorem allows us to find unbiased estimators with small variances.

Rao-Blackwell Theorem: Let $\hat{\theta}$ be an unbiased estimator of θ with $\text{var}(\hat{\theta}) < \infty$. Let U be a sufficient statistic for θ. Then define $\hat{\theta}^* = E(\hat{\theta} \mid U)$. Then:

Proof:

- The Rao-Blackwell Theorem says that if we have an unbiased estimator, we can get an unbiased estimator whose <u>variance</u> is <u>at least as small</u> by conditioning on a sufficient statistic.
- Which choice of sufficient statistic should we use?
- A <u>minimal</u> <u>sufficient</u> <u>statistic</u> is the sufficient statistic that "condenses the data" more than any other sufficient statistic.

<u>Definition</u>: A sufficient statistic is <u>minimal</u> <u>sufficient</u> if it is a function of every other sufficient statistic.

<u>Definition</u> (Completeness): A statistic U is <u>complete</u> if and only if

- <u>Note</u>: Any complete sufficient statistic is minimal sufficient.

- The Lehmann-Scheffé Theorem says that if we can construct an unbiased estimator that is a function of a ________ ________ ________, then this will be the Minimum Variance Unbiased Estimator (MVUE) of θ.

- Useful Result: If $Y_1, \ldots, Y_n$ is a random sample from a distribution in the one-parameter exponential family, i.e., having pdf of the form:

then:

- This result is useful for many examples!

Proof: (We prove the sufficiency of $\sum_{i=1}^{n} d(Y_i)$; the proof of completeness is quite advanced.)

- **Example 1:** $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Pois}(\lambda)$. Find the MVUE for λ.

- <u>Example 2</u>: $Y_1, \ldots, Y_n \overset{iid}{\sim}$ Weibull (θ), with $m = 2$ (see exercise 6.26). Find the MVUE for θ.

- The Lehmann-Scheffé theorem can also be used to find MVUEs for <u>functions of parameters</u> of some distribution.

- <u>Example 3:</u> $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Expon}(\beta)$. Find the MVUE for β^2 $(= \text{var}(Y_i))$.

- <u>Example 4</u>: $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu, 1)$. Find the MVUE for μ^2.

Example 5: Let $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Unif}(0, \theta)$.
Find a complete sufficient statistic for θ, and then find the MVUE of θ.

Note: The $\text{Unif}(0, \theta)$ distribution is not in the exponential family (its support ____________________). But we can show $U = Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$ is sufficient via the definition of sufficiency. We must show:

- We prove $U = Y_{(n)}$ is complete using the definition of completeness:

9.6 The Method of Moments

- The method of moments is a simple way to obtain point estimators.
- It involves equating population moments to sample moments and solving the equations.
- If we have to estimate t parameters, we set up t equations:

- Then solve these equations simultaneously for the desired parameters.

- Example 1: $Y_1, \dots, Y_n \overset{iid}{\sim} \text{Unif}(0, \theta)$. Find the MME of θ.

- <u>Example 2</u>: $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$. Find MMEs of α and β.

- Example 3: Let $Y_1, \dots, Y_n$ be iid r.v.'s with pdf

$$f_Y(y) = \begin{cases} \left(\frac{2}{\theta^2}\right)(\theta - y), & 0 < y < \theta \\ 0 & \text{elsewhere} \end{cases}$$

Find the MME of θ.

- Typically MMEs have low bias for large sample sizes.
- But... often MMEs are not functions of sufficient statistics, so they have poor efficiency relative to other possible estimators.